TOPOLOGY PRESENTATION 2

BEN WU

Theorem. *Let $\rho : X \to Y$ be a closed continuous surjective map such that $\rho^{-1}(\{y\})$ is compact, for each $y \in Y$. Show that if Y is compact, then X is compact.*

Proof. First, we claim that for any open set U containing $\rho^{-1}(\{y\})$, there exists a neighborhood W of y such that $\rho^{-1}(W)$ is contained in U. To see this, notice that since U is open, $X\backslash U$ is closed. Since ρ is a closed map, $\rho(X\backslash U)$ is also closed. It follows that $Y\backslash\rho(X\backslash U)$ is open. Set $W = Y\backslash\rho(X\backslash U)$. We claim that this is the neighborhood W we are looking for. To see this, notice that since $y \in \rho(U)$ and U contains $\rho^{-1}(\{y\})$, $y \notin \rho(X\backslash U)$. So $y \in W$ and W is a neighborhood about y. Now, consider $x \in \rho^{-1}(W)$. Then $\rho(x) \in W$. In particular, this implies that $\rho(x) \notin \rho(X\backslash U)$. So $\rho(x) \in \rho(U)$ and $x \in U$. So, W is a neighborhood of y such that $\rho^{-1}(W) \subseteq U$.

Now, let $\mathcal{A}$ be an open covering of X. For each $y \in Y$, $\mathcal{A}$ is an open covering of $\rho^{-1}(\{y\})$. Since $\rho^{-1}(\{y\})$ is compact, there exists a finite open subcover $A_1^y, \ldots, A_{m_y}^y$. Since $\rho^{-1}(\{y\})$ is contained in the open set $\cup_1^m A_i^y$, there exists a neighborhood W_y about y such that $\rho^{-1}(W_y) \subseteq \cup_1^m A_i^y$. Since we can find a W_y about each y, the collection $\{W_y | y \in Y\}$ is an open covering of Y. Since Y is compact, there exists a finite open subcover $\{W_{y1}, W_{y_2}, \ldots, W_{y_n}\}$. Since each $\rho^{-1}(W_{y_i})$ is covered by the finite collection $A_1^y, \ldots, A_{m_{y_i}}^{y_i}$ and there are finitely many $\rho^{-1}(W_{y_i})$'s, the collection $\{A_j^{y_i} | 1 \leq j \leq m_{y_i}, 1 \leq i \leq n\}$ is a finite open subcover of X.

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