Senran Lin **Presentation**

\$17.6 A, B, A_α are substes of X, prove:

1. $A \subset B \Rightarrow \overline{A} \subset \overline{B}$
2. $\overline{A \cup B} = \overline{A} \cup \overline{B}$
3. $\overline{\bigcup A_\alpha} \supset \bigcup \overline{A}_\alpha$

Proof

1. $A \subset B \Rightarrow \overline{A} \subset \overline{B}$
   $B \subset \overline{B} \Rightarrow A \subset B \subset \overline{B}$, and $\overline{B}$ is closed. Recall the definition of closure, $\overline{A}$ is the intersection of all closed sets containing A. Since $A \subset \overline{B}$ and $\overline{B}$ is closed, therefore $\overline{A} \subset \overline{B}$
2. $\overline{A \cup B} = \overline{A} \cup \overline{B}$

   - $\overline{A \cup B} \subset \overline{A} \cup \overline{B}$

     Since $A \subset \overline{A}$ and $B \subset \overline{B}$, then $A \cup B \subset \overline{A} \cup \overline{B}$ for $\forall x \in A \cup B$, $x \in \overline{A}$ or $x \in \overline{B}$. Since $\overline{A} \cup \overline{B}$ is a union of two closed sets, so $\overline{A} \cup \overline{B}$ is closed. Since $\overline{A} \cup \overline{B}$ is a closed set containing $A \cup B$, $\overline{A \cup B} \subset \overline{A} \cup \overline{B}$.
   - $\overline{A \cup B} \supset \overline{A} \cup \overline{B}$
     $A \subset A \cup B \Rightarrow \overline{A} \subset \overline{A \cup B}$ from (1). Similarly, $\overline{B} \subset \overline{A \cup B}$. Let $x \in \overline{A} \cup \overline{B}$, then $x \in \overline{A}$ or $x \in \overline{B}$. In either case, $x \in \overline{A \cup B}$. So $\overline{A \cup B} \supset \overline{A} \cup \overline{B}$.

   Therefore, $\overline{A \cup B} = \overline{A} \cup \overline{B}$.
3. $\overline{\bigcup A_\alpha} \supset \bigcup \overline{A}_\alpha$
   (The proof is analog to (2), here I provide another approach)
   Let $x \in \bigcup \overline{A}_\alpha$, then $\exists \overline{A_\alpha} : x \in \overline{A_\alpha}$. Then for any nbhd of x, U satisfies that U intersects $A_\alpha \Rightarrow U$ intersects $\bigcup A_\alpha$. By Theorem 17.5, $\overline{\bigcup A_\alpha} \supset \bigcup \overline{A}_\alpha$.
   To see why the equality does not hold, consider $A_i = \{1/i\}$ where $i \in \mathbb{Z}_+$. Then $\overline{\bigcup A_i} = \{1/i | i \in \mathbb{Z}_+\} \cup \{0\}$ And $\overline{A_i}$ is $\{1/i\}$ itself $\Rightarrow \bigcup \overline{A}_i = \{1/i | i \in \mathbb{Z}_+\}$. Therefore $\overline{\bigcup A_i} \not\subseteq \bigcup \overline{A}_i$