Math 432 – Topological Spaces **Homework 6**

Due April 6

For each problem below, please submit well-written, complete sentence solutions.
PROBLEM 1,2,4,5, or 7 will be on the quiz on Friday.

1. Prove that is X is an ordered set in which every closed interval is compact, then X has the least upper bound property.

2. Let $\mathbb{R}_K$ denote $\mathbb{R}$ with the $K-$topology.
   (a) Show that $[0,1]$ is not compact as a subspace of $\mathbb{R}_K$.
   (b) Show that $\mathbb{R}_K$ is connected.
   (c) Show that $\mathbb{R}_K$ is not path connected.

3. Let A_0 be the closed interval $[0,1]$ in $\mathbb{R}$. Let A_1 ne the set obtained from A_0 by deleting its "middle third" $(1/3, 2/3)$. Let A_2 be the set obtained from A_1 by deleting its "middle thirds" $(1/9, 2/9)$ and $(7/9, 8/9)$. In general, define A_n by the equation

$$A_n = A_n - 1 - \cup_{k=0}^{\infty}\left(\frac{1+3k}{3^n}, \frac{2+3k}{3^n}\right).$$

The intersection

$$C = \cap_{z\in\mathbb{Z}} A_n$$

is called the **Cantor set;** it is a subspace of $[0,1]$.
   (a) Show that C is totally disconnected (see page 152 #5).
   (b) Show that C is compact.
   (c) Show that each set A_n is a union of finitely many disjoint closed intervals of length $1/3^n$; and show that the end points of these intervals lie in C.
   (d) Show that C has no isolated points.
   (e) Conclude that C is uncountable.

4. Show that the rationals $\mathbb{Q}$ are not locally compact.

5. Let X be a locally compact space. If $f : X \to Y$ is continuous, does it follow that $f(X)$ is locally compact? What is f is both continuous and open? Justify your answer.

6. Show that $[0,1]^{\omega}$ is not locally compact in the uniform topology.

7. Show that the one point compactification of $\mathbb{Z}_+$ is homeomorphic with the subspace $\{0\}\cup\{1/n | n \in \mathbb{Z}_+\}$ of $\mathbb{R}$.

8. Read through examples 3, 4 and 5, in section 30. Write a short paragraph describing what you've read.

9. Show that if X has a countable basis $\{B_n\}$ then every basis $\mathcal{C}$ for X contains a countable basis for X.

10. Let X have a countable basis; let A be an uncountable subset of X Show that uncountably many points of A are limit points of A.

11. Let $f : X \to Y$ be a continuous open map. Show that if X satisfies the first or the second countability axiom, then $f(X)$ satisfies the same axiom.

12. Show that $\mathbb{R}^{\omega}$ in the uniform topology satisfies the first countability axiom but not the second.

13. Show that if X is regular, every pair of points of X have neighborhoods whose closures are disjoint.
14. Let $p : X \to Y$ be a closed continuous surjective map such that $p^{-1}\{y\}$ is compact for each $y \in Y$. (Such a map is called a *perfect map.*)
    (a) Show that is X is Hausdorff, then so is Y.
    (b) Show that is X is regular, then so is Y.
    (c) Show that is X is locally compact, then so is Y.
    (d) Show that is X is second-countable, then so is Y. (See hint in book).
15. (GRAD) Prove the following. You may need to reference section 22 which we did not cover.
    (a) If $p : X \to Y$ os a quotient map and if Z is a locally compact Hausdorff space, then the map
    $$\pi = p \times i_Z : X \times Z \to Y \times Z$$
    is a quotient map. (See hint in book).
    (b) Let $p : A\ toB$ and $q : C \to D$ be quotient maps. If B and C are locally compact Hausdorff spaces, then $p \times q : A \times C \to B \times D$ is a quotient map.
16. (GRAD) Give $\mathbb{R}^\omega$ the box topology. Let $\mathbb{Q}^\infty$ denote the subspace consisting of sequences of rationals that end in an infinite string of 0's. Which of our four countability axioms does this space satisfy?