Math 432 – Topological Spaces **Homework 7**

Due April 13

Homework Solutions

1. Show that every locally compact Hausdorff space is regular.

   Assume X is locally compact and Hausdorff. Then by Theorem 29.2 given any x in X and neighborhood U of x, there is a neighborhood V of x such that $\overline{V}$ is compact and $\overline{V} \subset U$. By Lemma 31.1, this is exactly what we need to to show X is regular.

2. Show that a connected normal space having more than one point is uncountable.

   Let X be a connected normal space with more than one point. Let x, y be two distinct points in X, since X is normal, it is T_1, so $\{x\}$ and $\{y\}$ are closed. By the Urysohn Lemma, there exists a continuous map $f : X \to [a, b]$ such that $f(x) = 0$ and $f(y) = 1$. Since X is connected, by the intermediate value theorem, the image of f must be the entire interval $[0, 1]$. And since we have a surjective function from X to an uncountable set, we can conclude that X is uncountable.

3. Show that a connected regular space having more than one point is uncountable. (Hint: Any countable space is Lindelöf.)

   Lemma Every regular Lindelöf space is normal.

   Proof of Lemma The proof works just like the one for Theorem 32.1 except that we don't have a countable basis. So we construct our covering of A by open sets whose closures do not intersect B using a similar technique and then use the Lindelöf property to make this a countable set. And do the same thing to get the countable cover of B. Then the proof continues the same.

   Proof of Statement Now, assume X is a connected regular space with at least two points but that X is countable. Since X is countable, X is Lindelöf, meaning every open cover has a countable subcover (this is easy to see - just take one set for each of the points in X). And thus X is normal, but this contradicts (a), X must be uncountable.

4. Let X be a compact Hausdorff space. Show that X is metrizable if and only if X has a countable basis.

   First assume X is a compact Hausdorff space with a countable basis. By theorem 32.3, X is normal, and thus regular so by the Urysohn metrization theorem, X is metrizable.

   Let X be a compact Hausdorff space that is metrizable. Let d be the metric. Consider for each $n \in \mathbb{Z}$, $\{B_d(x, 1/n) : x \in$X.$\}$ This is a cover of X and thus has a finite subcover we call $\mathcal{A}_n$. We will show that $\mathcal{A} = \cup_{n\in\mathbb{Z}}\mathcal{A}_n$ is a countable basis. First, its countable since it is the countable union of finite sets. We use Lemma 13.2 to show it is a basis. We know $\mathcal{A}$ is a collection of open sets of X. Let $U \subset X$ and $x \in U$. Let $\epsilon = d(x, X - U) > 0$. Find n so that $1/n < \epsilon/2$. We know that for every n, there exists y such that $x \in B_d(y, 1/n) \subset \mathcal{A}$. We just need to show that $B_d(y, 1/n) \subset U$. Assume not, i.e. there exists some point within $\epsilon/2$ of y and thus within ϵ of x, such that this point is not in U. Thus contradicts that $\epsilon = d(x, X - U)$. Thus, this is the countable basis we were seeking.

5. A collection $\mathcal{A}$ of subsets of X has the **countable intersection property** if every countable intersection of elements of $\mathcal{A}$ is nonempty. Show that X is a Lindelöf space if and only if for every collection $\mathcal{A}$ of subsets of X having the countable intersection property,

   $$\cap_{A\in\mathcal{A}}\overline{A}$$

   is nonempty.

We will show that every cover of X has a countable subcover if and only if there every collection of subsets with the countable intersection property has non-empty intersection.

Let X be a Lindelöf space. Let $\mathcal{A}$ be a collection of subsets of X having the countable intersection property. Let $\mathcal{C} = \{X - \overline{A} : A \in \mathcal{A}\}$. Then $\mathcal{C}$ is a collection of open sets. We have

$$\cup_{C \in \mathcal{C}} C = \cup_{A \in \mathcal{A}} (X - \overline{A}) = X - \cap_{A \in \mathcal{A}} \overline{A}.$$

We know $\cap_{A \in \mathcal{A}} \overline{A}$ is empty if and only if $\mathcal{C}$ is a cover of X. If we assume there is a countable subcover

$$X = \cup_{n \in \mathbb{Z}_+} C_n = \cup_{n \in \mathbb{Z}_+} X - \overline{A_n} = X - \cap_{n \in \mathbb{Z}_+} \overline{A}.$$

But by the countable intersection property, we know $\cap_{n \in \mathbb{Z}_+} \overline{A}$ is nonempty and thus we get a contradiction.

To go the other way, assume for every collection $\mathcal{A}$ of subsets of X having the countable intersection property,

$$\cap_{A \in \mathcal{A}} \overline{A}$$

is nonempty. Let $\mathcal{C}$ be an open cover of X. Assume $\mathcal{C}$ has no countable subcover. Thus for every countable collections of C_n, there exists some $x \in X - \cup_{n \in \mathbb{Z}} C_n = \cap_{n \in \mathbb{Z}} A_n$. But that means the A_n have the countable intersection property, which implies that $\cap_{A \in \mathcal{A}} A = X - \cup_{C \in \mathcal{C}} C$ is nonempty, a contradiction.