Math 432 – Topological Spaces
Presentation 2: Section 32 Exercise 4 **Ria Granzier**

Ex 4: Show that every regular Lindelof space is normal.

Proof: Let X be a regular Lindelof space. Let A and B be closed subsets of X. Since X is regular we know that for each $a \in A$ there exists an open neighborhood U_a of a such that $\bar{U_a} \subset X - B$. Similarly, for each $b \in B$ there exists an open neighborhood V_b of b such that $\bar{V_b} \subset X - A$. The set $\{U_a : a \in A\}$ covers A and $\{V_b : b \in B\}$ covers B. Also since A and B are closed subspaces of a Lindelof space they are Lindelof spaces themselves. Thus there exist countable subcovers of A and B, U and V, such that $U = \{U_1, U_2, ...\}$ and $V = \{V_1, V_2, ...\}$. Next define:
$W_1 = \{U_1 \cap \bar{V_1}^c\}$ $Z_1 = \{V_1 \cap \bar{U_1}^c\}$
$W_2 = \{U_2 \cap (\bar{V_1}^c \cap \bar{V_2}^c)\}$ $Z_2 = \{V_2 \cap (\bar{U_1}^c \cap \bar{U_2}^c)\}$
...
$W_k = \{U_k \cap (\bigcap_{i-1}^k \bar{V_i}^c)\}$ $Z_k = \{V_k \cap (\bigcap_{i-1}^k \bar{U_i}^c)\}$
...
Let $W = \bigcup_{i=1}^\infty W_i$ and $Z = \bigcup_{j=1}^\infty Z_j$. I claim that W and Z are open sets containing A and B such that W and Z are disjoint.
First note that all W_i and Z_j are open since they are finite intersections of open sets. So W and Z are open since they are unions of open sets. Next see that $U_i \cap A \subset W_i$ thus since U_i covers A, W_i cover A. So $A \subset W$ and similarly $B \subset Z$. Lastly I will show that $W_n \cap Z_m = \emptyset$ for some m, n. See that $W_n = U_n - (\bigcup_{i=1}^n \bar{V_i})$ so if $W_n \cap Z_m$ is nonempty then $m > n$. Similarly $Z_m = V_m - (\bigcup_{i=1}^m \bar{U_i})$ so if $W_n \cap Z_m$ is nonempty then $n > m$. Thus $W_n \cap Z_m = \emptyset$. So $W \cap Z = \emptyset$.
Thus W and Z are disjoint open sets which contain A and B. Therefore X is normal.