Presentation 2 - Topological Spaces - Math 432

Rafael Rojas

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Problem. *Let X be a metric space with metric d; let $A \subseteq X$ be nonempty.*

1. *Show that $d(x, A) = 0$ if and only if $x \in \bar{A}$.*

   Proof. To prove the statement we will show the inverse. That is, $d(x, A) > 0$ if and only if $x \notin \bar{A}$. For the forward direction, recall that $d(x, A) = inf\{d(x, a) \mid a \in A\}$. Since $d(x, A) > 0$, let $0 < \epsilon < d(x, A)$, then the open ball $B_d(x, \epsilon)$ is a neighborhood of x that is disjoint from A. Thus $x \notin \bar{A}$.

   Conversely, assume that $x \notin \bar{A}$. Then there is a neighborhood U of x such that $U \cap A = \emptyset$. Now choose ϵ such that $B_d(x, \epsilon) \subsetneq U$. Since $B_d(x, \epsilon) \cap A = \emptyset$, then it must be that $d(x, A) \geq \epsilon > 0$.

   □

2. *Show that if A is compact, $d(x, A) = d(x, a)$ for some $a \in A$.*

   Proof. The function $d : X \times X \to \mathbb{R}$ is continuous (by section 20, exercise 3). By restricting the function to the domain $x_0 \times A$ where x_0 is fixed we retain continuity. We can go a bit further and define the function $f(a) = d(x_0, a)$ for $a \in A$. This function is also continuous and maps A into $\mathbb{R}$. Since A is compact and $\mathbb{R}$ is an ordered set in the order topology, there exist $a_m, a_M \in A$ such that $f(a_m) \leq f(a) \leq f(a_M)$. But then $f(a_m) = inf\{d(x_0, a) \mid a \in A\}$. Therefore we have found that when A is compact, there does indeed exist an $a \in A$ such that $d(x, A) = d(x, a)$.

   □

Problem. *Is the space $\mathbb{R}_l$ connected?*

Proof. The claim is that the space $\mathbb{R}_l$ is not connected. The basis for the lower limit topology on $\mathbb{R}$ is the set of all elements of the form $[a, b)$. One of the ways we characterize the connectedness of a space is that it is connected if and only if the only sets that are both open and closed are the sets X and $\emptyset$. To show that $\mathbb{R}_l$ is not connected, consider the set $[0, 1)$. This set is open as it is a basis element. We know consider the complement of this set.

$$(-\infty, 0) \cup [1, \infty)$$

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These sets can be written in terms of the basis elements in the following way.

$$(\infty, 0) = \bigcup_{n \in \mathbb{Z}^+} [-n, 0)$$
$$[1, \infty) = \bigcup_{n \in \mathbb{Z}^+} [1, n + 1)$$

The equality of the sets can be shown by showing both the containments, but these are fairly clear so we won't do them here. Since both are arbitrary unions of open sets, they are open sets.

Therefore the complement of $[0, 1)$ is an open set, so $[0, 1)$ is also closed. But now we can use these to create a separation of $\mathbb{R}_l$ since

$$\mathbb{R}_l = [0, 1) \cup ((-\infty, 0) \cup [1, \infty))$$

and $\mathbb{R}_l$ is a union of disjoint, nonempty, open sets.

□