Presentation Ch.27, Problem 1 **Marcus Rosales**

If X is an ordered set in which every closed interval is compact, then X has the least upper bound property.

Proof: Assume X is an ordered set in which every closed interval is compact. Consider $A \subset X$ such that $A \neq \emptyset$, and let b be an upper bound on A. Consider $[a, b]$ for some $a \in A$; by our hypothesis $[a, b]$ is compact. Furthermore, we can see for:

$$C = \bar{A} \cap [a, b] \subset [a, b]$$

that C is a closed subset of a compact space; therefore, C is compact (Theorem 26.2). Consider the inclusion function:

$$i : C \to X \text{ defined by } i(z) = z \ \forall z \in C$$

By theorem 18.2.b. i is continuous, and since C is compact, and X is ordered, we are authorized to use the Extreme value theorem. In particular:

$$\exists \ c \in C \ \ni i(c) = c \geq z \ \forall z \in C$$

Now since $C \subset \bar{A}$, this implies $c \in \bar{A} = A \cup A'$; we have two cases to consider.

- $c \in A$, then clearly c is the least upper bound of A.
- $c \in A' - A$; that is c is a limit point of A. Suppose, by way of contradiction, that:

  $$\exists \ d \in C \ \ni \ \forall a_o \in A \ , a_o < d < c$$

  Note I didn't use a greater than or equal to symbol above since if $d = a_o$ for some $a_o \in A$, then d is our least upper bound like in the first case. Consider $\epsilon > 0$; notice $c \in (d, c + \epsilon)$. This is an open neighborhood about c, but notice:

  $$(A - \{c\}) \cap (d, c + \epsilon) = \emptyset$$

  This tells us that c is not a limit point of A; this can not be true! We conclude there is no such d that satisfies the above properties.

In both cases we see c is the least upper bound to our set A; thus, X has the least upper bound property.