THE COMPACTNESS THEOREM IN PROPOSITIONAL LOGIC

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The Compactness Theorem for Satisfiability considered here comes from Propositional Logic; it can be proved in a topological setting. The theorem states an equivalence, but one direction is trivial, so the other direction is given here and proved. A few definitions are needed to state the half-theorem.[1]

Definition. The set of *propositional variables* $\{A_1, A_2, \dots\}$ is a countable set.

Definition. A *truth assignment* is a function from the set of variables to $\{0, 1\}$.

Definition. A *formula* is a combination of propositional variables and the connectives $\{\wedge, \neg\}$ according to familiar formation rules (which are left unstated).

Definition. A formula is *satisfiable* if there is a truth assignment to its variables on which it is true (combining the truth of the variables in the familiar way for the usual semantics for 'and' ($\wedge$) and 'not' ($\neg$)). A set of formulas is satisfiable if each of its formulas is satisfiable.

Theorem. *Let $\mathcal{A} = \{A_1, A_2, \dots\}$ be a set of variables; let $\Phi = \{\phi_1, \phi_2, \dots\}$ be a set of formulas over $\mathcal{A}$. Then Φ is satisfiable if for all finite $\Phi_i \subset \Phi$, Φ_i is satisfiable.*

Proof. Let $F_\phi \subset \langle\{0,1\}\rangle^\omega$ be the set of assignments satisfying a formula ϕ.[2] Since ϕ has at most finitely many variables appearing in it, and only those variables determine ϕ's truth value, F_ϕ has finitely many factors not equal to $\{0,1\}$. By Theorem 19.1, then, F_ϕ is a basis element in the product topology, and is therefore open. Its complement is also open, however, so F_ϕ is closed in the product topology. An example of this setup is given in figure 0.1.

In the discrete topology $\{0,1\}$ is compact; by Tychonoff's theorem, $\langle\{0,1\}\rangle^\omega$ in the product topology is also compact. So by theorem 26.9,[3] it suffices to show that the collection of closed sets $F_\Phi = \{F_\phi \mid \phi \in \Phi\}$ has the finite intersection property. This is immediate from the hypothesis: The intersection $\bigcap_{\phi\in\Phi_i} F_\phi$ of a finite subcollection Φ_i of formulas contains exactly the assignments satisfying all formulas of Φ_i, so since by hypothesis Φ_i is satisfiable, $\bigcap_{\phi\in\Phi_i} F_\phi$ is nonempty, as desired. $\square$

[1] Note tuples are represented with angle brackets, as in $\langle a, b\rangle$.

[2] There is a one-to-one correspondence between truth assignments and omega tuples $\langle\{0,1\}\rangle^\omega$: for A the set of all truth assignments over $\mathcal{A}$, define $f : A \to \{0,1\}^\omega$ by $f(g) = \langle g(A_1), g(A_2), \dots\rangle$.

[3] **Theorem 26.9.** Let X be a topological space. Then X is compact iff for every collection $\mathcal{C}$ of closed sets in X having the finite intersection property, the intersection $\bigcap_{C\in\mathcal{C}} C$ of all elements of $\mathcal{C}$ is nonempty.

A_1	A_2	$A_1 \vee A_2$	$\neg A_2$
T	T	T	F
F	T	T	F
T	F	T	T
F	F	F	T

FIGURE 0.1. In the truth table, $\Phi = \{A_1 \vee A_2, \neg A_2\}$. Each row of the table gives part of a truth assignment, over the propositional variables appearing in Φ. Only truth assignments matching row three in their first two assignments are in F_Φ, after these two all possible values for the remaining tuple entries are included. Thus, for example $\langle 1, 0, 0, 0, \dots \rangle$, $\langle 1, 0, 1, 0, 0, \dots \rangle$, $\langle 1, 0, 1, 1, 0, 0, \dots \rangle$ are all included in F_Φ; more generally $F_\Phi = \{1\} \times \{0\} \times \{0, 1\} \times \{0, 1\} \times \cdots$. F_Φ is open and closed in $\langle \{0, 1\} \rangle^\omega$, since $\{1\} \times \{0\} \times \{0, 1\} \times \{0, 1\} \times \cdots$ is the complement of the open set $\{0\} \times \{1\} \times \{0, 1\} \times \{0, 1\} \times \cdots$.

References

[1] EBBINGHAUS, H.-D. *Mathematical logic*. Springer, 1994.

[2] TAO, T. The completeness and compactness theorems of first-order logic, Apr. 2009. https://terrytao.wordpress.com/2009/04/10/the-completeness-and-compactness-theorems-of-first-order-logic/.