Math 432 – Topological Spaces **Presentation 2**

Zach Schmidt

— A group G is called abelian if its operation commutes; $x * y = y * x$ for all $x, y \in G$.

— If a is a path in X from x_0 to x_1, then $\hat{a} : \pi_1(X, x_0) \to \pi_1(X, x_1)$ is defined by the equation $\hat{a}([f]) = [\overline{a}] * [f] * [a]$. $\hat{a}$ turns homotopy classes of loops based at x_0 into homotopy classes of loops based at x_1. By Theorem 52.1, $\hat{a}$ is an isomorphism between $\pi_1(X, x_0)$ and $\pi_1(X, x_1)$.

Section 52, exercise 3 (Munkres): Let x_0 and x_1 be points of the path-connected space X. $\pi_1(X, x_0)$ is abelian if and only if, for every pair a and b of paths from x_0 to x_1, $\hat{a} = \hat{b}$.

Proof: First, assume that $\pi_1(X, x_0)$ is abelian. Let a and b be paths from x_0 to x_1. I want to show that $\hat{a} = \hat{b}$. I'll do this by showing that $\hat{a}^{-1} = \hat{b}^{-1}$. (These inverses are defined because $\hat{a}$ and $\hat{b}$ are isomorphisms.) So consider some $[f] \in \pi_1(X, x_0)$. Note that $\hat{b}([f]) = [\overline{b}] * [f] * [b]$. It follows that

$$\hat{a}^{-1}(\hat{b}([f])) = [a] * [\overline{b}] * [f] * [b] * [\overline{a}].$$

Furthermore, $[a] * [\overline{b}] * [f]$ and $[b] * [\overline{a}]$ are loops based at x_0; they are in $\pi_1(X, x_0)$. So, since $\pi_1(X, x_0)$ is abelian and $\hat{a}^{-1}(\hat{b}([f])) = ([a] * [\overline{b}] * [f]) * ([b] * [\overline{a}])$, I can say that $\hat{a}^{-1}(\hat{b}([f])) = ([b] * [\overline{a}]) * ([a] * [\overline{b}] * [f]) = [f]$. Therefore, $\hat{a}^{-1} = \hat{b}^{-1}$, which means that $\hat{a} = \hat{b}$.

Now assume that $\hat{a} = \hat{b}$ for every pair a and b of paths from x_0 to x_1. Let $[f], [g] \in \pi_1(X, x_0)$. Choose a path a from x_0 to x_1. (X is path-connected.) Let $b = f * a$. Since $\hat{a}([f] * [g]) = \hat{b}([f] * [g])$,

$$\begin{aligned} \hat{a}([f] * [g]) &= [\overline{a}] * [\overline{f}] * [f] * [g] * [f] * [a] \\ &= [\overline{a}] * [g] * [f] * [a] \\ &= \hat{a}([g] * [f]). \end{aligned} \tag{1}$$

And $\hat{a}$ is an isomorphism, so $[f] * [g] = [g] * [f]$.