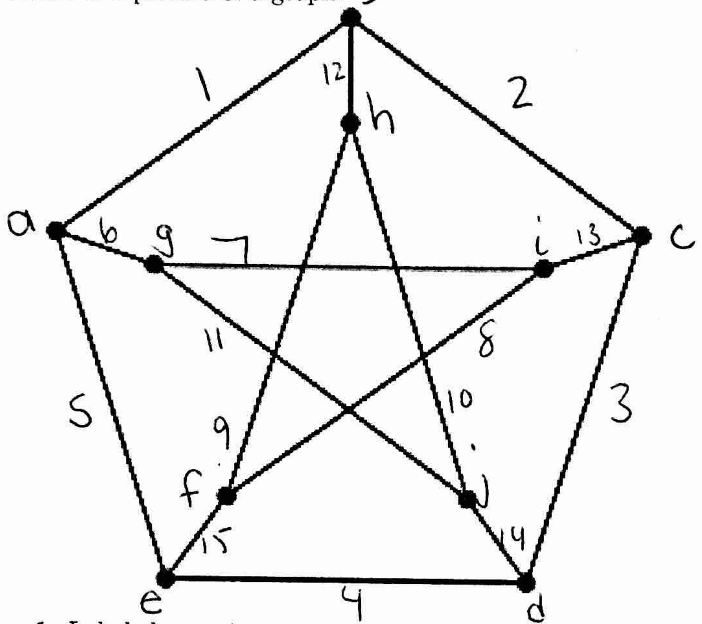
Section 3.2 Graphs and Digraphs*

Definitions

- A **graph** is a collection of vertices (points) and edges (lines between two points). The singular form of vertices is vertex.
- A **directed graph** or **digraph** is a graph whose edges have a defined direction, usually represented by an arrow.
- In a **digraph** the first vertex on a edge is called the **initial vertex**. The last vertex on a edge is called the **terminal vertex.** If we have an undirected graphs, we just call the vertices at the end of an edge the **endpoints.**
- We can have an edge which has the same vertex on each end. This edge is called a **loop**.
- We can have multiple edges between the same two vertices. There are called **parallel edges.** In a directed graph for two edges to be parallel, they must have the same initial vertices and terminal vertices (arrows must be going in the same direction).
- Two vertices are called **adjacent** if there is an edge between them.
- A **path** in a digraph G is a sequence of edges such that the terminal vertex of one edge is the initial vertex on the next. We can also have a path in an undirected graph then we don't have to worry about initial and terminal vertices, just endpoints. The **length** of a path is the number of edges in it.
- A path is **closed** is it starts and ends at the same vertex.
- A closed path of length at least 1 with vertex sequence $x_1x_2\cdots x_nx_1$ is called a **cycle** if $x_1,\ldots x_n$ are all different. and no repeated edges.
- A graph or digraph with no cycles is called **acyclic.** Some people call acyclic digraphs "**DAG**"s. (for directed acyclic graphs). I am not one of those people.
- A graph is **connected** is there is a path from every vertex to every other vertex.
- The **degree** of a vertex is the number of edges touching it.

*For Math 243:Katie Walsh

Below is a picture of a graph.



1. Label the vertices so that the vertex set is $\{a, b, c, \ldots j\}$
2. Label the edges so that the edge set are $\{1, 2, 3, \ldots 15\}$

answers may vary.

3. Is this graph connected?

Yes, there is a path from any vertex to any other vertex

4. Is this graph acyclic?

No, abciga is a cycle.

5. Give an example of a closed ~~loop~~ path which is not a cycle by listing the vertices in it.

eae (repeated edges)

agifhjga

repeated vertex

6. Which vertices are adjacent to vertex c?

b, i, d

answers depend on labeling

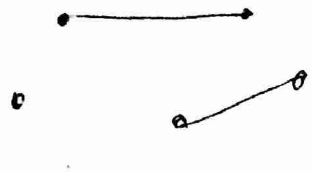
7. What is the degree of vertex a?

3 (true for every vertex).

Using the definitions on the first page, draw the following graphs and digraphs.

1. Draw a graph with 5 vertices and two edges. Is your graph connected?

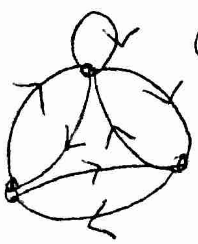
e.g.



not connected
impossible to be connected.

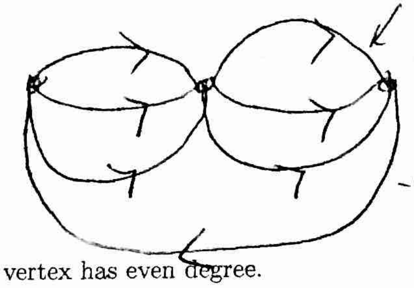
2. Draw a directed graph with 7 edges and 3 vertices. Does your graph have parallel edges?

e.g



✓ no parallel edges.

- or -

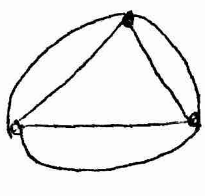


✓ parallel edges

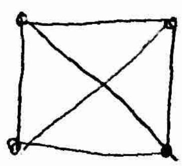
- or - many others

3. Draw a connected graph with 6 edges where every vertex has even degree.

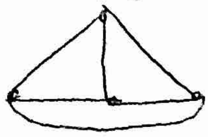
e.g:



4. Draw a graph with 4 vertices such that every vertex is connected to every other vertex by a path of length one. → means there is an edge between every pair of vertices.

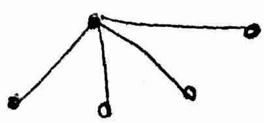


or



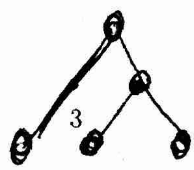
isomorphic

5. Draw a connected acyclic graph with 5 vertices. How many edges does it have?



or

or



all have 4 edges.

Matrices!

A matrix is a rectangular array of numbers. For example:

I start counting at 1.

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \end{bmatrix}$$

$$B = \begin{bmatrix} 2 & 3 \\ 7 & 2 \end{bmatrix}$$

$$C = \begin{bmatrix} 1 & 2 \\ 8 & 5 \end{bmatrix}$$

The size of a matrix is the number of rows by the number of columns. The first matrix has size 3×4. The other two are of size 2×2.

The i, j entry in a matrix in the entry in the ith row and the jth column. Given a matrix named M, we can also denote this as $M[i, j]$.

What is $A[2,3]$ for the matrix A given above?

7

Give a 3×4 matrix D such that $D[i, j] = i - j$.

$$\begin{bmatrix} 0 & -1 & -2 & -3 \\ 1 & 0 & -1 & -2 \\ 2 & 1 & 0 & -1 \end{bmatrix}$$

Given a matrix, you can multiply it by a scalar (just a regular number). To do this, you multiply every number in the matrix by that scalar.

Find $5B$ for the matrix B above.

$$\begin{bmatrix} 10 & 15 \\ 35 & 10 \end{bmatrix}$$

Given two matrices of the same size, you can add them (or subtract them) by just adding the corresponding entries. One way of writing this is:

$$(A + B)[i, j] = A[i, j] + B[i, j]$$

Write the above equations in words.

the i,j entry of A+B is same entry of A & B added together.

Find A+D

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 6 & 6 & 6 & 6 \\ 11 & 11 & 11 & 11 \end{bmatrix}$$

Find B-2C

$$\begin{bmatrix} 2 & 3 \\ 7 & 2 \end{bmatrix} - \begin{bmatrix} 2 & 4 \\ 16 & 10 \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ -9 & -8 \end{bmatrix}$$

We can also multiply two matrices of the same size. But we don't do it componentwise. We will talk about how to do that later.

Matrices and Graphs

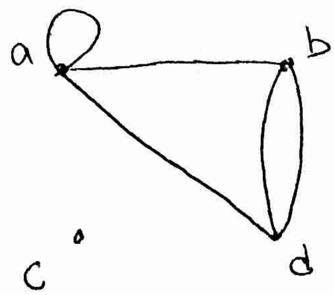
So, what's the relationship between matrices and graphs? Given a graph, we can record of the information about it (what the vertices and edges are) in a matrix. This matrix is called an **adjacency matrix.** How do we do this?

- Start with a square matrix ($n \times n$) where n is the number of vertices in the graph.
- For an edge between vertex a and vertex b, you put a 1 in row a column b. If there are multiple edges between the same vertex, put the number representing the number of edges.
- If you have a directed graph (digraph) you only put a number in row a and column b if a is the initial vertex and b in the terminal vertex.
- If you have an undirected graph, you for every edge from a to b you put a 1 (or more if there are parallel edges) in both the M[a,b] and M[b,a] spot.

For the matrices below, draw the corresponding graph or digraph.

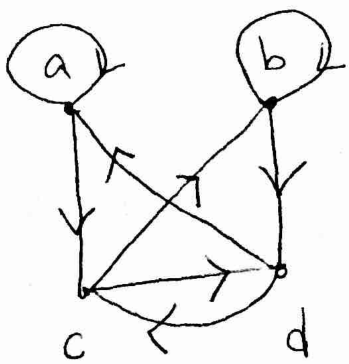
$$\begin{array}{cc} & \begin{array}{cccc} a & b & c & d \end{array} \\ \begin{array}{c} a \\ b \\ c \\ d \end{array} & \begin{bmatrix} 1 & 1 & 0 & 1 \\ 1 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 \\ 1 & 2 & 0 & 0 \end{bmatrix} \end{array}$$

Draw the graph:



$$\begin{array}{cc} & \begin{array}{cccc} a & b & c & d \end{array} \\ \begin{array}{c} a \\ b \\ c \\ d \end{array} & \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix} \end{array}$$

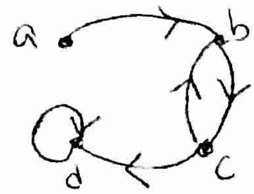
Draw the digraph:



The transpose of a matrix M, denoted M^T, is the matrix you get by changing all the rows of a matrix into columns. In particular

$$M^T[i,j] = M[j,i]$$

Draw a digraph of your choice with at least 4 vertices and 5 edges.



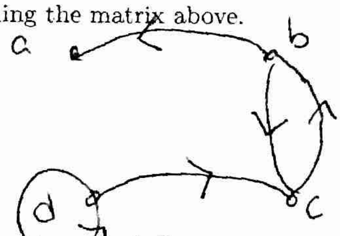
Find the adjacency matrix of the graph you drew.

$$M = \begin{array}{c} \\ a \\ b \\ c \\ d \end{array} \begin{array}{c} \begin{array}{cccc} a & b & c & d \end{array} \\ \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \end{array}$$

Find the transpose of the adjacency matrix of the graph you drew.

$$M^T = \begin{array}{c} \\ a \\ b \\ c \\ d \end{array} \begin{array}{c} \begin{array}{cccc} a & b & c & d \end{array} \\ \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix} \end{array}$$

Draw the digraph corresponding the matrix above.



What do you notice about your two graphs?

Same, except arrows are reversed.

What happens when you take the transpose of an adjacency matrix of an undirected graph? symmetric

In an undirected graph the matrix is ~~symmetric~~
So taking transpose doesn't change it.

One final definition: A matrix M is called **symmetric** if $M = M^T$. The adjacency matrix of an undirected graph is always symmetric.