Section 1.4 Set Operations

Sometimes, we want to combine sets in special ways to get new sets. Here are some example sets we'll use in this section.

$$S = \{a, 2, b, 4\}$$

$$T = \{a, b, c, d, e\}$$

$$W = \{x \in \mathbb{N} : x \text{ is even}\}$$

.

The two most common set operations are union and intersection. Look at the definitions below and see if they agree with your ideas of what union and intersection should be.

$$\text{union: } A \cup B = \{x : x \in A \text{ or } x \in B\}$$

$$\text{intersection: } A \cap B = \{x : x \in A \text{ and } x \in B\}$$

Draw two Venn Diagrams for general overlapping sets A and B and color in the union on one and the intersection on the other.

Find $S \cup T, T \cup W, S \cup W$.

Find $S \cap T, T \cap W, S \cap W$

In math, "or" always includes "or both". Sometimes, we might want to find all the elements in A or B but not both. This is called the symmetric difference denoted $\oplus$

$$A \oplus B = \{x : x \in A \text{ or } x \in B \text{ but not both }\}$$

We can also find the set of all elements in A that are not in B. This is called the relative complement and denoted $A \backslash B$ or $A - B$.

Write a definition for $A \oplus B$ using only $\cup, \cap$, and $\setminus$.

Draw Venn Diagrams for $A \setminus B$ and $A \oplus B$.

Write down three different statements each using at least three symbols $(\cup, \cap, \oplus, \setminus)$ and three sets (S, T, W or the number sets we talked about). Then trade with your groupmates and have them determine what your statements are equal to.

Let $A = \{a, b, c, d\}$. What are all the elements not in A?

Did you remember "matrices, graphs and non-mathematical objects such as stars, ideas, snails, and puppy dogs' tails." (I have to give the book credit for that line). When we talk about the "elements not in a set," we usually think of elements from some certain set. We call this overall set the **universe** or **universal set.** Once we have a universal set defined, then we can talk about the **absolute complement** or **complement** of A, denoted A^C and is equal to $U \backslash A$.

Show via a Venn Diagram that

$$A \backslash B = A \cap B^C$$

Draw a Venn Diagram for $A^C \cap B^C$.

Draw a Venn Diagram for $(A \cup B)^C$.

What do you notice about the last two diagrams? This illustrates the fact that

$$(A \cup B)^C = A^C \cap B^C$$

Draw a Venn Diagram to illustrate the fact that

$$(A \cap B)^C = A^C \cup B^C$$

The last two laws are called DeMorgan's Laws. Write what these laws mean in words.

Draw Venn Diagrams for:

$$(A \cap B) \cup (A \cap C) \qquad (A \cup B) \cap (A \cup C) \qquad A \cup (B \cap C) \qquad A \cap (B \cup C)$$

Which sets are equal?

These are called the distribution laws. Can you see why?

There are some other algebra laws for sets but they are pretty intuitive. Read through the list below and fill in any blanks.

1. $A \cup B = B \cup A$ commutative law for union
2. $A \cap B =$ ________ commutative law for intersection
3. $(A \cup B) \cup C = A \cup (B \cup C)$ associative law for union
4. ____________ = ____________ associative law for intersection
5. $A \cup A =$ ____ idempotent law for union
6. $A \cap A =$ ____ idempotent law for intersection
7. $A \cup \emptyset =$ ____ identity law
8. $A \cup U =$ ____ identity law
9. $A \cap \emptyset =$ ____ identity law
10. $A \cap U =$ ____ identity law
11. $(A^C)^C =$ ____ double complementation
12. $A \cup A^C =$ ____
13. $A \cap A^C =$ ____
14. $U^C =$ ____
15. $\emptyset^C =$ ____

Show (using Venn Diagrams) that the symmetric difference is associative.

Given two sets, A and$B$,we can define a new set, $A \times B$, which is the set of ordered pairs of the form (a, b) where $a \in A$ and $b \in B$. Here, order is important. One special example of this is $\mathbb{R} \times \mathbb{R} = \mathbb{R}^2$, which is the cartesian plane.

Let $A = \{1, 2\}$ and $B = \{a, b, c\}$. Then $A \times B$ is

We can also draw a picture of $A \times B$ by writing all elements of A as labels to columns, and all of the elements B as labels to rows and then just drawing markers for each element. Below is a picture of $A \times B$.

c	*	*
b	*	*
a	*	*
	1	2

Let S={1,2}. Write out the elements of $S^2 = S \times S$ and draw a picture of this set.

Be careful! (1,2) is an elements of the set above - it's an ordered pair. But sometimes we write (1,2) to mean the open interval $\{x \in \mathbb{R} : 1 < x < 2\}$. You just have to tell from context which is the one we are talking about.

You showed the distribution laws and DeMorgan's laws using Venn Diagrams. But what if you wanted to show them in words? To show that two sets A and B are equal, you can just show that

$$A \subseteq B$$

and

$$B \subseteq A$$

And to show one set is a subset of another, you have to show (going back to the definition) that every element (or an arbitrary element) in the first set is also in the second set. To give you a sense of what these proofs look like, I've written the proof of one of the distribution laws below. Read through it with your groupmates and see if it all makes sense and if you can see why this counts as a proof. I won't ask you to prove things like this on your own...yet.

To prove

$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C),$$

we will show that $A \cup (B \cap C) \subseteq (A \cup B) \cap (A \cup C)$ and $A \cup (B \cap C) \supseteq (A \cup B) \cap (A \cup C)$.

First, let $x \in A \cup (B \cap C)$. Then by definition, either $x \in A$ or $x \in B \cap C$. If $x \in A$, then $x \in A \cup B$ and $x \in A \cup C$, therefore, $x \in (A \cup B) \cap (A \cup C)$. In the other case, if $x \in B \cap C$, then $x \in B$ and $x \in C$ so $x \in A \cup B$ and $x \in A \cup C$, therefore, $x \in (A \cup B) \cap (A \cup C)$. In either case, $x \in A \cup (B \cap C)$ implies $x \in (A \cup B) \cap (A \cup C)$. Therefore

$$A \cup (B \cap C) \subseteq (A \cup B) \cap (A \cup C).$$

Now, let $x \in (A \cup B) \cap (A \cup C)$. Then $x \in A \cup B$ and $x \in A \cup C$. Since either x is in A or not, we will consider two cases, either $x \in A$ or $x \notin A$. If $x \in A$ then $x \in A \cup (B \cap C)$. If $x \notin A$ since we have assumed $x \in A \cup B$ and $x \in A \cup C$, this means that $x \in B$ and $x \in C$ and thus $x \in B \cap C$ and $x \in A \cup (B \cap C)$. Therefore

$$A \cup (B \cap C) \supseteq (A \cup B) \cap (A \cup C).$$

Finally, since $A \cup (B \cap C) \subseteq (A \cup B) \cap (A \cup C)$ and $A \cup (B \cap C) \supseteq (A \cup B) \cap (A \cup C), A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$.