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Lecture - ~~14~~ + 15

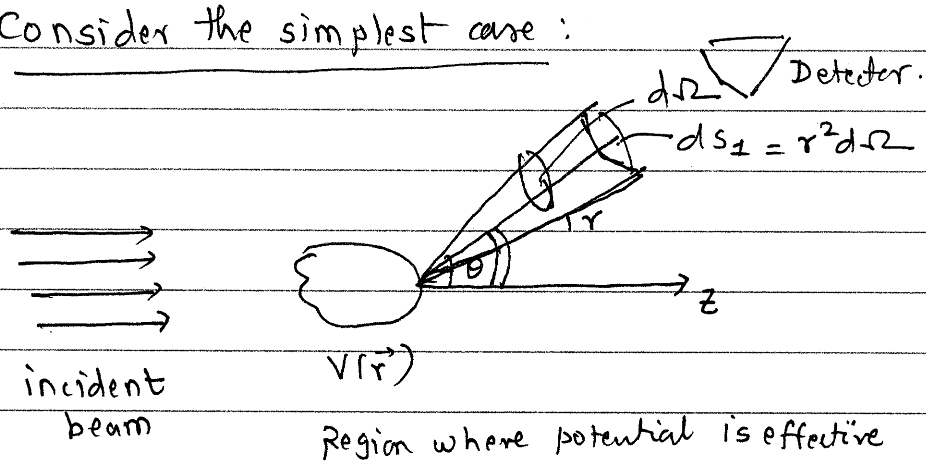
Scattering Theory :

Refs : Cohen-Tanoudji 901 - 936, Chapter VIII
Merzbacher Chapter 11 214 - 235
Sakurai 379 - 392, 407 - 422
Schiff 114 - 129

Why is scattering important?

Almost all the information on structure of matter, beginning from the existence of nucleus in an atom in Rutherford scattering expt. or classically we know how we get information about the potential from scattering experiments.

Consider the simplest case :



We consider the problem

$$1 + 2 \rightarrow 1 + 2$$

Elastic scattering - i.e. none of the particles change their character or transfer any energy between projectile & target.
Neglect the spin for the time being, in centre of mass frame of 1 + 2

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~~Consider the~~ Let m_1, m_2 be the masses of the two particles.

The reduced mass of the system

$$\frac{1}{\mu} = \frac{1}{m_1} + \frac{1}{m_2}$$

The two body central force can ~~is~~ then be reduced to motion of a single particle and in terms of $\vec{r} = |\vec{r}_1 - \vec{r}_2|$; $V \equiv V(\vec{r})$.

For an electron in an atom $\frac{1}{\mu} \approx \frac{1}{m_e}$

Definition of scattering cross-section

F_i : Flux of the incident particles, i.e. number of particles incident per unit time per unit area ; dimension $[L^{-2}T^{-1}]$

dn : no. of particles scattered into solid angle $d\Omega$ around θ, ϕ. per unit time

$dn \propto F_i \cdot d\Omega$.

~~The proportionality~~ ~~factor~~ ~~constant can dep~~

The proportionality factor can depend on θ, ϕ, say $f(\theta,\phi)$. ~~call this~~

$$dn = F_i\, \sigma(\theta,\phi)\, d\Omega \Rightarrow \boxed{\frac{1}{F_i}\frac{dn}{d\Omega} = \sigma(\theta,\phi)}$$

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Dimension of $\sigma(\theta,\phi)$ is $L^2 \equiv$ area

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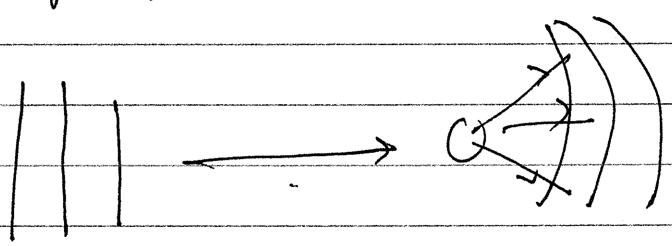
$\therefore$ Number of particles reaching the detector per unit time is equal to number of particles crossing a surface $\sigma(\theta,\phi)\,d\Omega$.

$\sigma(\theta,\phi)$; differential cross-section. $\equiv d\sigma/d\Omega$

In the end we want to learn how to calculate $\sigma(\theta,\phi)$ given $V(\vec{r})$. ~~i.e.~~ we want to now deal with states in the continuum. Given a potential, the spectrum can consist of bound states + continuum. In scattering experiments, the incident energy can be chosen arbitarily. Hence we are necessarily solving Schrödinger Eq^n. for the continuous part of energy spectrum.



scattering of a wave packet

We want to compute intensities of scattered beam at a given point in space.

If size of wave packet >> scattering region $\Rightarrow$ the calculation should be indep. of the specifics of the wave packet.

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We want to study time evolution of the wave packet

In case the above approximation holds, we can analyse it in terms of plane waves.

In particular, we want to ~~solve~~ analyse Eigenstates of a Hamiltonian

$$H = H_0 + V(\vec{r}) \qquad \text{where } H_0 = \frac{\vec{p}^{\,2}}{2m} \left(= \frac{\vec{p}^{\,2}}{2\mu} \right)$$

Classical scattering problem :

	~~Bound~~ closed orbits	open orbits
	$\Updownarrow$	$\Updownarrow$
Q. M. Scatt. problem	stationary states Eigenstates with discrete Energy E values	Scattering stationary states (Continuum)

The problem to solve is to obtain solutions with appropriate asymptotic behaviour.

Take the example of a potential barrier.

$$\begin{aligned} V(x) &= 0 & x < 0 & \quad \text{I} \\ &= V_0 & 0 < x < a & \quad \text{II} \\ &= 0 & x \geq a. & \quad \text{III} \end{aligned}$$

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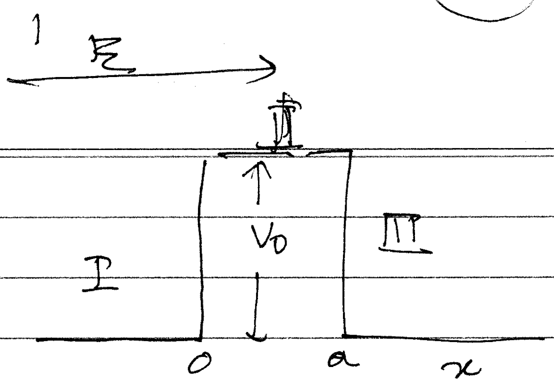

$$\psi(x) = A e^{ikx} + B e^{-ikx} \qquad x < 0$$
$$= C e^{ikx} \qquad x \geq a$$

For $E > V_0$ we have ~~two~~ ~~plan~~ plane wave solutions in both regions I & III. But the boundary conditions, imposed so that they have ~~proper~~ correct asymptotic behavior tells us the correct choice.

For example in region I we will have both an incoming and outgoing wave (travelling towards right & left) where in region III we have only the transmitted ~~a~~ wave $\propto e^{ikx}$ travelling towards right, : This is for ~~an~~ a particle incident from the left. In 1d this problem is simple and tractable for ~~a~~ the case of a piecewise continuous potential and we can solve A, B & C in terms of V_0, a

For example, for $E > V_0$ we get resonant transmission for

$$\sqrt{2m(E-V_0)}\ \frac{a}{\hbar} = \pi,\ 2\pi, -- \quad \cancel{(\text{HW})}$$

Whereas we also get tunneling for $E < V_0$.

How to choose the solutions with appropriate boundary conditions in ~~3d~~ three dimensional problems

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Let us now look at 3d problem. Consider the solution to Schrödinger Eq^n. given by corresponding to energy Evalue E

$$\boxed{\Psi(\vec{r},t) = \phi(\vec{r})\, e^{-iEt/\hbar}} \quad -(87)$$

~~$\Phi(\vec{r},t)$~~

Equation satisfied by $\phi(\vec{r},t)$ is than

$$\left[-\frac{\hbar^2}{2\mu}\vec{\nabla}^2 + V(\vec{r})\right]\phi(\vec{r}) = E\,\phi(\vec{r}) \;;\; E = \frac{\hbar^2 k^2}{2\mu}$$

$$\therefore \boxed{\left[\vec{\nabla}^2 - \frac{2\mu V(\vec{r})}{\hbar^2} + k^2\right]\phi(\vec{r}) = 0} \quad -(88)$$

* ~~In the regions of no potential the plane $e^{i\vec{k}\cdot\vec{r}}$ solution~~

For the incident wave (in the region where $V(\vec{r})=0$), incident along z axis, the correct solution is e^{ikz}. What about the solution in region far away from the scattering centre, to the right.

Incoming wave: plane wave.

Scattered wave, i.e. the solution of the Schrödinger Eq^n after scattering, as a result of the ~~action of pot~~ effect of the potential: $\phi(\vec{r})$ which are used in describing the scattering process, which have the right boundary conditions are called $\boxed{\text{Stationary Scattering states}}$. They satisfy Eq. (88).

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~~Consider the~~

Recall solution of the three dimensional problem using separation of variables ~~for~~ a central potential.

~~For large r, $V(\vec{r}) \to 0$~~

~~$\psi(\vec{r}), \phi(\vec{r}) = R$~~

$$\phi(\vec{r}) = R(r)\, Y_l^m(\hat{\theta}, \phi) \qquad \text{with} \quad R(r) = \frac{u(r)}{r}$$

Use Eq. (88) to show that this means that $u(r)$ satisfies

$$-\frac{\hbar^2}{2\mu}\frac{d^2u}{dr^2} + \left[\frac{\hbar^2}{2\mu r^2} l(l+1) + V(r)\right] u(r) = E\, u(r)$$

Two things : (1) $u(r) \to 0$ as $r \to 0$ ~~for~~ the only acceptable sol^n ~~...~~ for the full eq^n. including ~~in the region~~ region at $r=0$

(2) For $r \to \infty$, $V(r) = 0$, ~~R(r)~~ $u(r) \propto e^{ikr}$

$$R(r) \underset{r\to\infty}{\sim} \frac{e^{ikr}}{r}$$

[For $r \to \infty$ this is acceptable]

$\hookrightarrow$ outgoing spherical wave. We are choosing the sol^n with right ~~boundary~~ asymptotic behaviour.

$$\boxed{\therefore \phi_{sc}(\vec{r})\Big|_{r\to\infty} \propto \frac{e^{ikr}}{r}}$$

$\Rightarrow$ The scattered state is

$$\boxed{\phi_{sc}(\vec{r}) \sim f_k(\theta,\phi)\frac{e^{ikr}}{r}} \quad \to (89)$$

~~$\Rightarrow$ Solution of Eq^n~~ Wave

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~~Another way of seeing the same thing~~:

~~Consider the~~

~~% The~~
Note also that $\boxed{(\vec{\nabla}^2 + k^2)\frac{e^{ikr}}{r} = 0 \quad \text{for } r \geq r_0 \text{ for } r_0 > 0}$

$\Rightarrow \frac{e^{ikr}}{r}$ is an outgoing wave with the same energy as the incoming wave.

$\therefore$ For the stationary asymptotic states we must have

$$\boxed{\phi(\vec{r}) \underset{r\to\infty}{\sim} e^{ikz} + f_k(\theta,\phi)\frac{e^{ikr}}{r}} \qquad (90)$$

$$\boxed{(\vec{\nabla}^2 + k^2)\,\phi(\vec{r}) \neq 0 \qquad \text{for } r \to \infty}$$

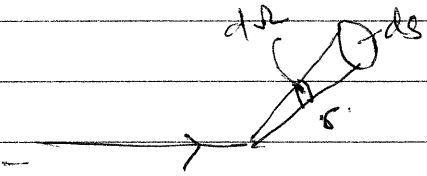
$\boxed{f_k(\theta,\phi) \text{ is called the scattering amplitude}}$

Relationship between $f_K(\theta,\phi)$ & $d\sigma/d\Omega$

Start with $\phi(\vec{r})$ at $r \to \infty$.

∴ Far from the scattering centre we have :

$$\boxed{\phi(\vec{r}) \simeq e^{ikz} + f_k(\theta,\phi).\frac{e^{ikr}}{r}} \quad \text{(90)}$$



$$ds = r^2 d\Omega$$

Prob. Current $\vec{J}(\vec{r}) = \frac{1}{\mu} Re\left[\phi^*(\vec{r}) \frac{\hbar}{i} \vec{\nabla}\phi(\vec{r})\right] \longrightarrow$ (90')

Incident wave $\propto e^{ikz} \Rightarrow |\vec{J}_i| = \frac{\hbar K}{\mu}$

number of particles incident per unit area per unit time

$$\boxed{F_i = C|\vec{J}_i| = \frac{C\hbar K}{\mu}} \quad \text{(91)} \qquad \dim[F] = L^{-2}.T^{-1}$$

Velocity of particles

Scattered current in the asymptotic region [i.e. one corresponding to $\phi_{sc}(\vec{r})$]

$$\cancel{(\vec{J}_{sc})_{\hat{r}}} \quad \cancel{(\vec{J}_{sc})_{r\,comp.}} = \cancel{\frac{\hbar K}{\mu}} \cancel{\frac{1}{r}} \cancel{|f(K)|}$$

Recall $(\vec{\nabla})_{\hat{r}} = \frac{\partial}{\partial r}$ (Recall expansion of $\vec{\nabla}$ in sp. polar coordinates)

Use Eqn. (90')

$$\Rightarrow \boxed{(\vec{J}_{sc})_{\hat{r}} = \frac{\hbar K}{\mu} \frac{1}{r^2} |f_k(\theta,\phi)|^2} \quad \text{(92)}$$

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dn = no. of particles crossing area $r^2 d\Omega$

$$= c\, \vec{J}_{sc} \cdot d\vec{s} = c r^2 d\Omega \cdot \frac{\hbar k}{\mu} \cdot \frac{1}{r^2} |f_k(\theta,\phi)|^2$$

$$\boxed{dn = c \cdot d\Omega \cdot \left(\frac{\hbar k}{\mu}\right) |f_k(\theta,\phi)|^2} \longrightarrow (93)$$

Eq. 86 $\quad \boxed{dn = d\Omega\, F_i\, \sigma(\theta,\phi)}$

Using (93) & (86) and (91) we have

$$d\Omega\, F_i\, \sigma(\theta,\phi) = c\, d\Omega\, \frac{\hbar k}{\mu} |f_k(\theta,\phi)|^2$$

$$\cancel{d\Omega}\, \cancel{c} \frac{\hbar k}{\mu}\, \sigma(\theta,\phi) = \cancel{c}\, \cancel{d\Omega}\, \frac{\hbar k}{\mu} |f_k(\theta,\phi)|^2$$

$$\boxed{\sigma(\theta,\phi) = |f_k(\theta,\phi)|^2 \longrightarrow (94)} \equiv \frac{d\sigma}{d\Omega}$$

differential cross-section

$$\boxed{\frac{d\sigma}{d\Omega} = |f_k(\theta,\phi)|^2}$$

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Alternate presentation of scattering cross-section. (Not to be discussed in class)

$$\underset{k}{\phi}(\vec{r}) = A\left[e^{ikz} + f_k(\theta,\phi)\frac{e^{ikr}}{r}\right] \qquad A \sim L^{-3/2}$$

Prob. per unit time that the scattered particle will pass thro' an area ~~$\Delta \vec{s}$~~ element $d\vec{s}$ is given by

$$|A|^2 \underset{\text{velocity}}{c} \frac{|f_k(\theta,\phi)|^2}{\cancel{r^2}}\, d\Omega \cdot \cancel{r^2} \;\text{≈ dim}\; T^{-1} \rightarrow \boxed{\text{I}}$$

~~Prob.~~ The incident particle density is $|A^2|c$. dim $L^{-2}T^{-1}$ $\boxed{\text{II}}$

$$d\sigma = \frac{|A|^2 \cancel{c}\, |f_k(\theta,\phi)|^2 d\Omega}{|A|^2 \cancel{c}} = f_k(\theta,\phi)|^2\, 2\pi \sin\theta\, d\theta \rightarrow \boxed{\text{III}}$$

$\boxed{\text{III}}$ is ratio of I & II and has dimensions L^2

i.e. area. This is called the differential cross-section.

- Probability to find the scattered particles ~~in ar~~ passing through area $d\vec{s}$ ~~is~~ per unit time, per unit incident flux is called the differential cross-section $d\sigma \Rightarrow \boxed{\frac{d\sigma}{d\Omega} = |f_k(\theta,\phi)|^2}$ (V)

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Aim of scattering theory :

1) To calculate $f_K(\theta, \phi)$ for a given potential $V(\vec{r})$.

2) To do that we need to set up a mathematical formalism, to calculate $f_K(\theta, \phi)$.

Two aspects : energy dependence & angular dependence.

convenient to discuss it in different angular momentum channels for a spherically symm. potential

a) Born Approximation b) Partial wave expansion.

Repeat

$d\sigma$: ~~differential cross section~~

= probability of scattered particles to pass through an area $d\vec{s} = r^2 d\Omega \, \hat{r}$ per unit time, per unit incident flux

$= d\Omega \, |f_K(\theta, \phi)|^2 = d\Omega \, \sigma(\theta, \phi)$