Integration by Substitution (Review)

Part 1: Compute the indefinite integral in three steps: (1) specify the substitution $u =$? and $du =$?, (2) rewrite the integral *completely* in terms of the new variable u, (3) compute the new integral and express the final answer in terms of the original variable.

Example. Evaluate $\displaystyle\int \frac{x}{x^2+1}\,dx$.

Solution.

(1) Set $u = x^2 + 1$, so $du = 2x\,dx$.

(2) Since $x\,dx = \frac{1}{2}du$, we have $\displaystyle\int \frac{x}{x^2+1}dx = \int \frac{(1/2)du}{u} = \int \frac{1}{2}\cdot\frac{1}{u}\,du$.

(3) $\displaystyle\int \frac{1}{2}\cdot\frac{1}{u}\,du = \frac{1}{2}\ln|u| + C = \frac{1}{2}\ln|x^2+1| + C = \frac{1}{2}\ln(x^2+1) + C$.

1. Evaluate $\displaystyle\int x^2\sin(x^3)\,dx$.

Let $u = x^3$, $du = 3x^2\,dx$

$$\Rightarrow \int x^2\sin(x^3)\,dx = \int \sin(u)\,\frac{du}{3} = -\frac{\cos(u)}{3}$$

$$= \boxed{-\frac{\cos(x^3)}{3} + C}$$

2. Evaluate $\displaystyle\int x\sqrt{4x+1}\,dx$.

Let $u = 4x+1$, $du = 4\,dx$

$\Rightarrow x = \dfrac{u-1}{4}$

$$\int x\sqrt{4x+1}\,dx = \int \frac{1}{4}(u-1)\sqrt{u}\,\frac{du}{4} = \frac{1}{16}\int u^{3/2} - u^{1/2}\,du$$

$$= \frac{1}{16}\left[\frac{2}{5}u^{5/2} - \frac{2}{3}u^{3/2}\right] = \boxed{\frac{1}{16}\left[\frac{2}{5}(4x+1)^{5/2} - \frac{2}{3}(4x+1)^{3/2}\right] + C}$$

3. Evaluate $\int \frac{1+x}{1-x}\,dx$.

Let $u = 1-x$, $du = -dx$ $\Rightarrow$ $x = 1-u$

$$\int \frac{1+x}{1-x}\,dx = \int \frac{1+(1-u)}{u}\,\frac{du}{-1} = \int \frac{u-2}{u}\,du = \int 1 - \frac{2}{u}\,du$$

$$= u - 2\ln|u| = \boxed{1-x-2\ln|1-x|+C}$$

4. Evaluate $\int \frac{1}{x\ln x}\,dx$.

Let $u = \ln x$, $du = \frac{1}{x}\,dx$

$$\int \frac{1}{x\ln x}\,dx = \int \frac{1}{u}\,du = \ln|u| = \boxed{\ln(\ln x) + C}$$

5. Evaluate $\int \sin^3 x\,dx$. (Hint: $\sin^2 x = 1 - \cos^2 x$.)

$$\int \sin^3 x\,dx = \int \sin^2 x \cdot \sin x\,dx = \int (1-\cos^2 x)\sin x\,dx$$

Let $u = \cos x$, $du = -\sin x\,dx$ $\Rightarrow$

$$= \int (1-u^2)\,\frac{du}{-1} = \int u^2 - 1\,du = \frac{u^3}{3} - u$$

$$= \boxed{\frac{\cos^3 x}{3} - \cos x + C}$$

Part 2: Express the definite integral in x as a new definite integral in the new variable u. Don't evaluate the new definite integral.

Example. Rewrite $\int_2^3 xe^{-x^2}\,dx$ in terms of $u = x^2$.

Solution. When $u = x^2$, $du = 2x\,dx$. If $x = 2$ then $u = 4$, and if $x = 3$ then $u = 9$, so $\int_2^3 xe^{-x^2}\,dx = \int_{u=4}^{u=9} e^{-u}\frac{du}{2} = \frac{1}{2}\int_4^9 e^{-u}\,du.$

6. Rewrite $\int_0^1 (3x+1)^2\,dx$ in terms of $u = 3x+1$. $\Rightarrow du = 3\,dx$

$u(0) = 3(0)+1 = 1$
$u(1) = 3(1)+1 = 4$ $\Rightarrow \int_1^4 u^2 \frac{du}{3}$

7. Rewrite $\int_0^1 x^2(1+2x^3)^5\,dx$ in terms of $u = 1+2x^3$. $\Rightarrow du = 6x^2\,dx$

$u(0) = 1+2(0)^3 = 1$
$u(1) = 1+2(1)^3 = 3$ $\Rightarrow \int_1^3 u^5 \frac{du}{6}$

8. Rewrite $\int_0^{\pi/3} \frac{\sin x}{\cos^2 x}\,dx$ in terms of $u = \cos x$. $\Rightarrow du = -\sin x\,dx$

$u(0) = \cos 0 = 1$
$u(\pi/3) = \cos\frac{\pi}{3} = \frac{1}{2}$ $\Rightarrow \int_1^{1/2} \frac{-1}{u^2}\,du$

9. Rewrite $\int_0^{\pi/3} \sin x \cos x\,dx$ in terms of $u = \cos x$. $\Rightarrow du = -\sin x\,dx$

$\int_1^{1/2} -u\,du$

10. Rewrite $\int_0^{\pi/3} \sin x \cos x\,dx$ in terms of $u = \sin x$. $\Rightarrow du = \cos x\,dx$

$u(0) = \sin 0 = 0$
$u(\pi/3) = \sin\pi/3 = \frac{\sqrt{3}}{2}$ $\Rightarrow \int_0^{\frac{\sqrt{3}}{2}} u\,du$

Math 1132 Worksheet 3 **Due**: 2nd discussion section in the week of Feb. 17th

Methods of Integration

Solutions to these problems should show all of your work, not just a single final answer.

Part 1: Integration by parts. Do each problem as follows: (1) specify u and dv, (2) compute du and v, (3) use integration by parts with your choice of u and dv. (4) If you need integration by parts more than once, each time go through steps 1, 2, and 3 again.

Example. Compute $\int x^2 e^x \, dx$.

Solution.

(1) Set $u = x^2$ and $dv = e^x \, dx$.

(2) We have $du = 2x \, dx$ and $v = e^x$.

(3) Now $\int x^2 e^x \, dx = \int u \, dv = uv - \int v \, du = x^2 e^x - \int e^x (2x) \, dx = x^2 e^x - 2 \int x e^x \, dx$.

(4) To find $\int x e^x \, dx$, set $u = x$ and $dv = e^x \, dx$, so $du = dx$ and $v = e^x$. Then $\int x e^x \, dx = \int u \, dv = uv - \int v \, du = x e^x - \int e^x \, dx = x e^x - e^x$.

(5) Substituting the result of (4) into (3),

$$\int x^2 e^x \, dx = x^2 e^x - 2(x e^x - e^x) + C = (x^2 - 2x + 2)e^x + C.$$

1. Compute $\int x \cos(5x) \, dx$.

$u = x$ $\quad dv = \cos(5x)\,dx$

$du = dx$ $\quad V = \dfrac{\sin(5x)}{5}$

$$\Rightarrow \frac{x \sin(5x)}{5} - \int \frac{\sin(5x)}{5} \, dx = \boxed{\frac{x \sin(5x)}{5} + \frac{\cos(5x)}{25} + C}$$

2. Compute $\int x^2 2^x \, dx$. (Hint: You can find an antiderivative of 2^x by recalling how to differentiate 2^x.)

u		dv
x^2	$+$	2^x
$2x$	$-$	$\frac{2^x}{\ln 2}$
2	$+$	$\frac{2^x}{(\ln 2)^2}$
0		$\frac{2^x}{(\ln 2)^3}$

$$\Rightarrow \boxed{\frac{x^2 2^x}{\ln 2} - \frac{2x\, 2^x}{(\ln 2)^2} + \frac{2 \cdot 2^x}{(\ln 2)^3} + C}$$

Part 2: Integration of rational functions.

Example. Compute $\int \frac{2x+1}{x^2-4}\,dx$ using partial fractions.

Solution. Write $\frac{2x+1}{x^2-4} = \frac{A}{x+2} + \frac{B}{x-2}$ for some A and B. Clearing the denominator, $2x+1 = A(x-2)+B(x+2)$. Setting $x=2$ we get $5=4B$, so $B=5/4$. Setting $x=-2$ we get $-3=-4A$, so $A=3/4$. Thus $\frac{2x+1}{x^2-4} = \frac{3/4}{x+2} + \frac{5/4}{x-2}$, so

$$\int \frac{2x+1}{x^2-4}\,dx = \int \left(\frac{3/4}{x+2} + \frac{5/4}{x-2}\right) dx = \frac{3}{5}\ln|x+2| + \frac{5}{4}\ln|x-2| + C.$$

3. Compute $\int \frac{10}{x^3-x^2-6x}\,dx$ using partial fractions.

$$= \int \frac{A}{X} + \frac{B}{X-3} + \frac{C}{X+2}\,dx$$

$$= \int \frac{-5/3}{X} + \frac{2/3}{X-3} + \frac{1}{X+2}\,dx$$

$$= \boxed{-\frac{5}{3}\ln|X| + \frac{2}{3}\ln|X-3| + \ln|X+2| + C}$$

$10 = A(X-3)(X+2) + BX(X+2) + CX(X-3)$

$X=0$: $10 = -6A \Rightarrow A = -5/3$

$X=3$: $10 = 15B \Rightarrow B = 2/3$

$X=-2$: $10 = 10C \Rightarrow C = 1$

4. Compute $\int \frac{x^2+x+1}{x(x^2+4)}\,dx$ using partial fractions.

$x^2+X+1 = A(x^2+4) + (Bx+C)X$

$= (A+B)X^2 + CX + 4A$

$\Rightarrow C=1$, $1=4A \Rightarrow A = \frac{1}{4}$

$A+B=1 \Rightarrow B = \frac{3}{4}$.

$$= \int \frac{A}{X} + \frac{Bx+C}{X^2+4}\,dx = \int \frac{\frac{1}{4}}{X} + \frac{\frac{3}{4}X+1}{X^2+4}\,dx$$

$$= \int \frac{\frac{1}{4}}{X} + \frac{\frac{3}{4}X}{X^2+4} + \frac{1}{X^2+4}\,dx$$

$$= \int \frac{\frac{1}{4}}{X} + \frac{\frac{3}{8}\cdot 2X}{X^2+4} + \frac{1}{4}\frac{1}{(\frac{X}{2})^2+1}\,dx = \boxed{\frac{1}{4}\ln|X| + \frac{3}{8}\ln(X^2+4) + \frac{1}{2}\tan^{-1}\left(\frac{X}{2}\right) + C}$$