Two-Way ANOVA

- Two factors [a]: X_1 has I levels and X_2 has J levels.
- Data y_{ijl} can be displayed in a two-way table with I rows and J columns. The (i, j)th cell contains n_{ij} obs

$$y_{ij1}, \ \ldots, \ y_{ijn_{ij}}.$$

- A balanced design has $n_{ij} = m$.

[a] A factor is a categorical predictor with possible values called levels.

Possible Models

- The interaction model (the most general model) $\mathcal{M}_{R\times C}$

$$y_{ijl} = \mu + \alpha_i + \beta_j + \gamma_{ij} + e_{ijl}.$$

- The additive model $\mathcal{M}_{R+C}$

$$y_{ijl} = \mu + \alpha_i + \beta_j + e_{ijl}.$$

- The row-effect model $\mathcal{M}_R$ $\quad y_{ijl} = \mu + \alpha_i + e_{ijl}.$
- The column-effect model $\mathcal{M}_C$ $\quad y_{ijl} = \mu + \beta_j + e_{ijl}.$
- The intercept-only model $\mathcal{M}_0$ $\quad y_{ijl} = \mu + e_{ijl}.$

The LS Estimates

- We can write the (element-wise) models in matrix form.
- Yes, some are over-parameterized. There are multiple ways to reduce parameters, .e.g., we can remove some columns. But no matter which approach we take, the resulting LS estimates should be the same.

- $\mathcal{M}_{R\times C}$ $\hat{y}_{ijl} = \bar{y}_{ij\cdot}$
- $\mathcal{M}_{R+C}$ $\hat{y}_{ijl} = \bar{y}_{i\cdot\cdot} + \bar{y}_{\cdot j\cdot} - \bar{y}_{\cdot\cdot\cdot}$ [a]
- $\mathcal{M}_{R}$ $\hat{y}_{ijl} = \bar{y}_{i\cdot\cdot}$
- $\mathcal{M}_{C}$ $\hat{y}_{ijl} = \bar{y}_{\cdot j\cdot}$
- $\mathcal{M}_{0}$ $\hat{y}_{ijl} = \bar{y}_{\cdot\cdot\cdot}$

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[a] This is true only for balanced design. For unbalanced design, the LS for $\mathcal{M}_{R+C}$ does not have a closed-form expression.

Which model to pick?

- Recall the partial F-test for comparing two nested models:

$$H_0 : \text{a smaller model with } p_0 \text{ coefficients}$$

$$H_a : \text{a large model with } p_a \text{ coefficients}$$

$$\begin{aligned} F &= \frac{\left(\text{RSS}_0 - \text{RSS}_a\right)/(p_a - p_0)}{\hat{\sigma}_a^2} \\ &\sim F_{p_a - p_0, n - p_a} \text{ under the null.} \end{aligned}$$

- Suppose we are comparing three nested models:

$$\mathcal{M}_1 \subset \mathcal{M}_2 \subset \mathcal{M}_3 \text{ with dim } p_1 < p_2 < p_3.$$

- We make our decision through the following path:

  1) Compare $\mathcal{M}_2$ vs $\mathcal{M}_3$. If $\mathcal{M}_3$ is selected, stop;

  2) otherwise, compare $\mathcal{M}_1$ vs $\mathcal{M}_2$.

- When calculating the F-stat at step 2, we could use $\hat{\sigma}^2$ from $\mathcal{M}_2$, i.e., the F-test $\sim F_{p_2-p_1, n-p_2}$ or we could calculate the two F-stats simultaneously using $\hat{\sigma}^2$ from $\mathcal{M}_3$ (the largest model), so the F-test for the comparison at step 2 would be $F_{p_2-p_1, n-p_3}$.

- Back to the two-way ANOVA model. Due to the hierarchical structure, we make our decision through the following path:

  1) Compare $\mathcal{M}_{R\times C}$ vs $\mathcal{M}_{R+C}$. If select the interaction model, stop; otherwise, go to the next step.

  2) Pick one among the remaining four models,

$$\mathcal{M}_{R+C}, \quad \mathcal{M}_{R}, \quad \mathcal{M}_{C}, \quad \mathcal{M}_{0},$$

  depending on whether the row or column effect is significant.

  Step 2 is much easier for the balanced design.

The Balanced Design ($n_{ij} = m > 1$)

	SS(Sum Sq)	df
Interaction	RSS($\mathcal{M}_{R+C}$) - RSS($\mathcal{M}_{R\times C}$)	$(I-1)(J-1)$
Row	RSS($\mathcal{M}_0$) - RSS($\mathcal{M}_R$)	$I-1$
Col	RSS($\mathcal{M}_0$) - RSS($\mathcal{M}_C$)	$J-1$
Err	RSS($\mathcal{M}_{R\times C}$)	$n-IJ$

Three F-tests: SS/df is the numerator, and the dominator is SS/df, i.e., $\hat{\sigma}^2$ from the interaction model

$$\hat{\sigma}^2 = \frac{\text{RSS}(\mathcal{M}_{R\times C})}{(m-1)IJ}.$$

Selected Model	Interaction	Row	Col
$\mathcal{M}_{R\times C}$	Sig	—	—
$\mathcal{M}_{R+C}$	No	Sig	Sig
$\mathcal{M}_{R}$	No	Sig	No
$\mathcal{M}_{C}$	No	No	Sig
$\mathcal{M}_{0}$	No	No	No

The Balanced Design ($n_{ij} = m = 1$)

- Only one observation in each cell, so we cannot fit the interaction model.
- RSS($\mathcal{M}_{R\times C}$) $= 0$, i.e., the corresponding error variance is 0.
- Then consider $\mathcal{M}_{R+C}$, instead of $\mathcal{M}_{R\times C}$, to be the largest model. All the F-tests are the same except that the interaction model is not a candidate model.

The Unbalanced Design

- Compare $\mathcal{M}_{R\times C}$ vs $\mathcal{M}_{R+C}$: if the F-test is significant, stop.
- If the interaction is not significant, we need to pick one model from

$$\mathcal{M}_{R+C}, \quad \mathcal{M}_R, \quad \mathcal{M}_C, \quad \mathcal{M}_0.$$

- The difficulty: to decide whether the column effect is significant, we can

  a) compare $\mathcal{M}_{R+C}$ vs $\mathcal{M}_R$ (test the column effect given that the row effect has been included), or

  b) compare $\mathcal{M}_C$ vs $\mathcal{M}_0$ (test the column effect given that the row effect is not included).

- For a balanced design, these two tests turn out to be the same, but that's not the case for the unbalanced design.
- In most cases, it is not difficult to make a consensus decision.
- Let's take a look of the two ANOVA tables.

	SS(Sum Sq)	df
C	RSS($\mathcal{M}_0$) - RSS($\mathcal{M}_C$)	$J-1$
$R \mid C$	RSS($\mathcal{M}_C$) - RSS($\mathcal{M}_{R+C}$)	$I-1$
Interaction	RSS($\mathcal{M}_{R+C}$) - RSS($\mathcal{M}_{R\times C}$)	$(I-1)(J-1)$
Err	RSS($\mathcal{M}_{R\times C}$)	$n-IJ$
TSS	RSS($\mathcal{M}_0$)	$n-1$

	SS(Sum Sq)	df
R	RSS($\mathcal{M}_0$) - RSS($\mathcal{M}_R$)	$I-1$
$C \mid R$	RSS($\mathcal{M}_R$) - RSS($\mathcal{M}_{R+C}$)	$J-1$
Interaction	RSS($\mathcal{M}_{R+C}$) - RSS($\mathcal{M}_{R\times C}$)	$(I-1)(J-1)$
Err	RSS($\mathcal{M}_{R\times C}$)	$n-IJ$
TSS	RSS($\mathcal{M}_0$)	$n-1$

Selected Model	R	$R \mid C$	C	$C \mid R$
$\mathcal{M}_0$	$\times$	$\times$	$\times$	$\times$
$\mathcal{M}_{R+C}$	—	$\surd$	—	$\surd$
	$\surd$	—	—	$\surd$
	—	$\surd$	$\surd$	—
$\mathcal{M}_R$	$\surd$	$\surd$	—	$\times$
	$\surd$	$\times$	$\times$	$\times$
$\mathcal{M}_C$	—	$\times$	$\surd$	$\surd$
	$\times$	$\times$	$\times$	$\surd$
$\mathcal{M}_{R+C}$	$\times$	$\surd$	$\times$	$\times$
	$\times$	$\times$	$\times$	$\surd$

	SS(Sum Sq)	df	F-stat
R	20	10	$2 \sim F_{10,50}$
$C \mid R$	16	8	$2 \sim F_{8,50}$
C	11	8	$1.375 \sim F_{8,50}$
$R \mid C$	25	10	$2.5 \sim F_{10,50}$
$R + C$	36	18	$2 \sim F_{18,50}$
Err	50	50	$\hat{\sigma}^2 = 1$