IISc, Fundamentals of Astrophysics, PH 217

Solution of Problem Set I

September 7, 2016

1. **What is a sidereal day and a synodic day ? The former is equivalent to one full rotation of the earth relative to fixed stars and the latter is relative to the Sun. How are the two related ? A synodic day is 24 hours. How long is the sidereal day ?**

   A day is in some sense a measure of the rotation rate of the earth. It is measured by tracking the apparent motion of other objects as seen from the earth. A synodic day (or a solar day) is defined as the time between two consecutive passages of the sun over the local meridian. Thus, it is related to the apparent motion of the sun in the sky. A sidereal day is defined as the time between two consecutive passages of distant stars. As Distant stars are fixed in the sky (except for effects like precession and nutation - see Prob. 5), the sidereal day gives a more accurate measure of the rotation rate than the synodic day. These two days can be related by considering the fact that the sun moves once around the earth (with respect to the distant stars) in a year. Thus, there would be one lesser synodic day per year than sidereal day. Thus

   $$t_{sidereal} = \frac{365.242}{366.242} \times t_{synodic}$$

   For a 24 hour synodic day, this gives $t_{sidereal} \sim$ 23 hours 56 min 4.1 s.

2. **What is the bolometric apparent magnitude of the Sun? Its absolute bolometric magnitude is 4.8. Schematically plot the SED (spectral energy distribution) of the Sun, an O-star and an M-star on the same plot (log-log plot is a good choice). Schematically show the bolometric correction for the three cases. Recall that the sun's spectrum peaks in V band. What about the B-V color index?**

   Apparent and absolute magnitudes are related by the expression

   $$m - M = 2.5\, log_{10} \left(\frac{d}{10kpc}\right)^2$$

   For sun's distance of 1 A.U $\sim$ 1/206265 pc, we get its apparent magnitude to be -26.77.

   The plot of the SED for three stars of O-type (T = 41000 K), M-type (T = 2700 K) and the sun (T = 5800 K) is plotted assuming the SED to be a black-body. It is shown in Fig.

. As seen from the figure, the V-band is near the peak of the sun like star and far off from the peaks of the M and O stars. Thus the bolometric correction (w.r.t to the V band) for both these stars are very high as compared to the sun. The calculation of bolometric correction for these stars follows from the definition of the absolute bolometric magnitude for any star which is given as

$$M_{bol} = -2.5log_{10}\left(\frac{L_{star}}{L_{sun}}\right) + M_{bol,\odot}(= 4.8)$$

For a star emitting like a black-body $L_{star} \propto R_*^2 T_*^4$ The bolometric correction is $BC = M_{bol} - M_V$, where M_V is the absolute magnitude of the star as measured in the V band. This will depend on the integrated flux in the window band of the V filter, which changes as seen in Fig. for different stars.

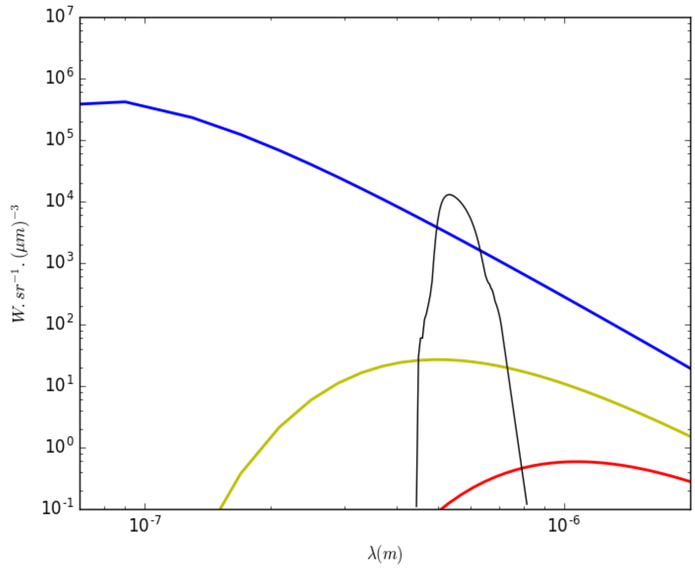

Figure 1: SED of O-star (blue), M-star (red) and the sun(yellow) along with the filter transmission of V-band (black)

3. **Because of a finite speed of light (c), the light that we see now was emitted earlier, corresponding to the light travel time from the source to the observer. It is 8 minutes for the Sun-Earth distance. How long ago the light that we see now from Proxima Centauri emitted? What about the light emitted from the center of the Milky Way?**

The distances to Proxima Centauri and the center of the Milky way are roughly 1.3 pc and 8 kpc away from earth, giving light travel times of 4.2 years and 26 thousand years respectively.

4. **Hubble Space Telescope has a primary mirror of diameter 2.4 m. How closely spaced bright point objects in angle can it resolve observing at 0.5 microns? Express the result in arc-seconds. How does the resolving power of the human eye compare to this (assume a pupil diameter of 0.5 cm)? People are contemplating a 30 m telescope (TMT). How much better will its resolution be compared to HST? Recall that this can only be achieved by adaptive optics that can compensate for much larger angular distortions due to**

the atmosphere (seeing). What is the resolution achievable by using VLBI spread over the globe at radio (say 200 GHz) wavebands? People are trying to image the Galactic center black hole using sub-mm interferometry (see: `http://www.eventhorizontelescope.org`).

The resolving power for diffraction limited telescopes is given by $r = 1.22\frac{\lambda}{D}$, where D is the telescope diameter and λ the wavelength of observation. Thus for HST (2.4m at 0.5 microns) r = 0.05 arcsec, for human eye (0.5cm at 0.5 microns) r = 25 arc-sec, for VLBI (2×6400 kms at 200 GHz) r = 30 μ arc-sec.

5. **What is meant by precession of equinoxes? Why does it happen? Would this occur if the earth's rotation axis were not tilted relative to the ecliptic plane or if the earth was not oblate? Why? Estimate the precession time period to an order of magnitude. Find out about the age of Aries, Pisces, and Aquarius and their relation to precession.**

If we observe the night sky then we can see that the whole sky appears to rotate around a point (star). This indicates the rotation of earth about its axis. In northern hemisphere, there is a bright star called Polaris (belongs to the constellation Ursa Minor) is situated along that direction. In southern hemisphere, at a similar location, the name of the star is Sigma Octantis (belongs to the constellation Octans). These points are known as north celestial pole (NCP) and south celestial pole (SCP) respectively which are believed to be the part of a imaginary sphere called celestial sphere (see Fig. 2). If the earth's axis of rotation is always fixed then the direction will never be changing with time. However, it is observed that, in Northern hemisphere, it slowly ($\dot{\phi} \approx 7.65 \times 10^{-12}$ rad s^{-1}) rotates about the constellation Draco (North Ecliptic pole, NEP). This is called the precession of the earth's axis of rotation.

Let us now imagine a plane perpendicular to the earth's axis of rotation and passes through the center of earth (i.e., passes through the earth's equator). This plane is called celestial equator. The angle between celestial equator and the ecliptic plane (the orbital plane of earth and sun system) is 23.5^0. The locus of intersection between these two planes is a straight line which cuts the celestial sphere at two points (as viewed from earth), see Fig. 2. These two points are called equinoxes. Due to the precession of the earth's axis, the celestial plane also rotates which changes the position of the equinoxes. This is called precession of equinoxes.

Derivation :

Here we will consider a *simple* toy model to know the physics behind the precession, and at the end we will estimate the time period for the precession of earth's axis (or equinoxes).

- **Tidal force per unit mass:** The gravitational acceleration due to an object of mass $M_\bullet$ at a distance d (at point c in Fig. 3) is

$$\vec{a}(d) = \frac{G\,M_\bullet}{d^2}\hat{y} \tag{1}$$

where $G = 6.673\,\mathrm{g}^{-1}\,\mathrm{cm}^3\,\mathrm{s}^{-2}$ is the universal gravitational constant. Let us consider an another point at a distance $d+r$ (point A3) along the same direction but $r << d$. The acceleration at this point is

$$\vec{a}(d+r) = \frac{G\,M_\bullet}{(d+r)^2}\hat{y} \tag{2}$$

The relative acceleration (w.r.t. point c in Fig. 3) between these points is

$$\Delta a = \vec{a}(d+r) - \vec{a}(d) \approx -\frac{2\,G\,M_\bullet r}{d^3}\;\hat{y} \tag{3}$$

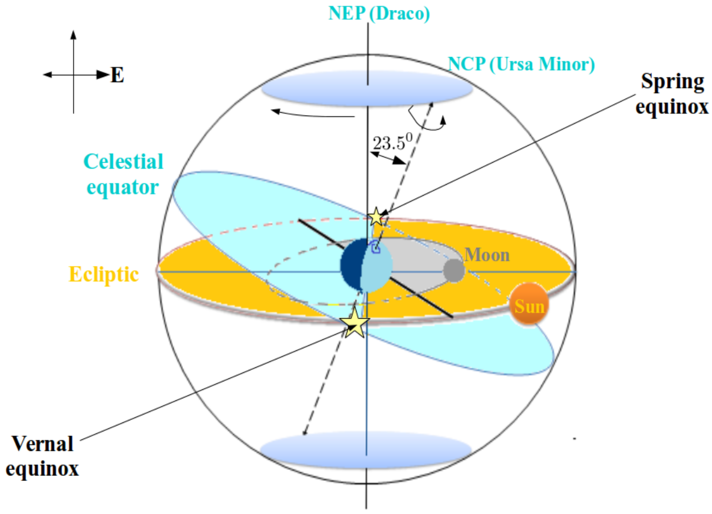

Figure 2: Geocentric universe. The dashed line passing through the center of earth is the earth's axis of rotation. The plane perpendicular to the earth's axis and passing through the center of earth is called celestial equator. The plane containing the earth's orbit around the sun is called ecliptic plane. The angle between equatorial plane and ecliptic is 23.5^0. The locus of intersection between these two planes is a straight line and two extreme points are called equinoxes (marked by the star). If the earth's axis is fixed then these two points always remain along the same direction. If the earth's axis precesses then the equatorial plane also precesses, and as a result, the equinoxes changes their location.

This differential force per unit mass is known as the tidal acceleration (a_{tidal}). In a similar way, we can find the tidal acceleration at distance $d-r$ (point A1): $\vec{a}_{\text{tidal}} = \frac{2\,G\,M_\bullet r}{d^3}\,\hat{y}$.

A more generation derivation shows that the tidal force at a point located in $y-z$ plane and which makes an angle θ w.r.t. the line A3-c-A1 is

$$\vec{a}_{\text{tidal}} \simeq \frac{G\,M_\bullet\,r}{d^3}(2\,\cos\theta\hat{y} - \sin\theta\hat{z}) \tag{4}$$

- **Tidal force on bulge :** Due to the axial rotation, the shape of the earth is slightly oblate rather than a perfect sphere. The equatorial radius is $R_c \approx 6,371\,\text{km}$ and polar radius $R_p \approx 6353\,\text{km}$. So, we can estimate the excess of mass (we may call it as the bulge mass) due to the deviation from a perfect sphere as

$$m_{\text{bulge}} = \left[\frac{4\pi}{3}R_p\,R_c^2\right]\rho_{\text{mean}} - \left[\frac{4\pi}{3}R_{\text{p}}^3\right]\rho_{\text{mean}} \approx \left[1 - \left(\frac{R_p}{R_c}\right)^2\right]M_\oplus \simeq 0.0097\,M_\oplus \tag{5}$$

where $M_\oplus$ is the total mass of the earth. Therefore, the tidal force on the bugle is $\vec{F}_{\text{tidal}} = m_{\text{bulge}}\,\vec{a}_{\text{tidal}}$

Earth's rotational frequency ($2\pi/24\,\text{hrs}$) is approximately 29 times than the moon's orbital frequency around the earth. So, if we consider the bulge as a single test object

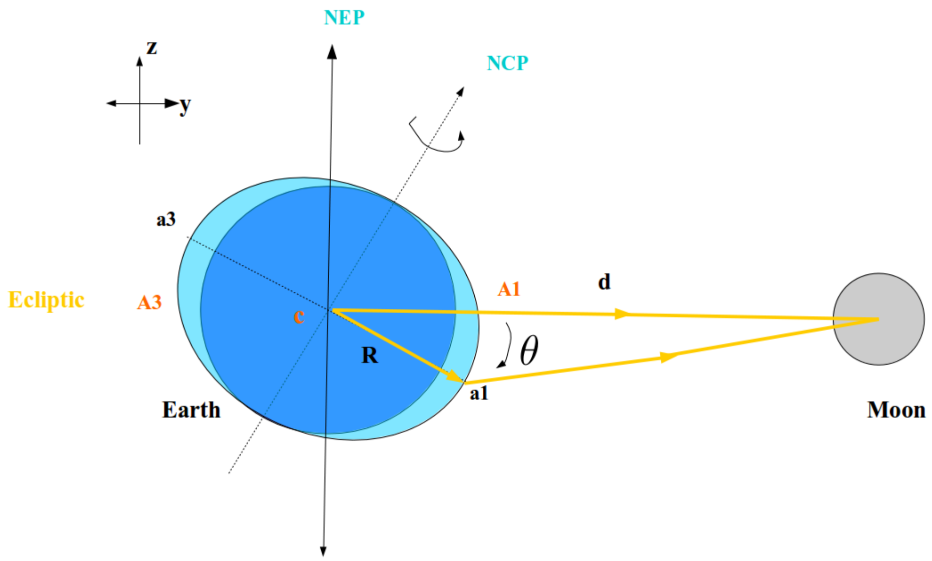

Figure 3: Schematic diagram of earth-moon system. In this figure, we have considered the plane of the paper as $y - z$ plane and the position of moon is along the positive y direction.

of mass $m_{\rm bulge}$, then from moon it may look like a continuous object distributed near the earth's equator.

Let us now consider the bugle mass is currently located in y-z plane at $a1$ (see Fig. 3) and the position vector of the bulge is : $\vec{r} = R_{\oplus}(\cos\theta\hat{y} - \sin\theta\hat{z})$ ($r \simeq R_{\oplus}$ is the mean radius of the earth). The tidal torque at this position is :

$$\vec{\tau}_{\rm tidal} = \vec{r} \times \vec{F}_{\rm tidal} = \frac{GM_{\bullet}\, m_{\rm bulge} R_{\oplus}^2}{d^3} \cos\theta \sin\theta\ \hat{x} \tag{6}$$

Therefore, at point a1 in Fig. 3, the direction of tidal force is pointing out of this paper (see Fig. 3). This means, the tidal torque pulls the axis to precess clockwise if it is viewed from the positive z-axis. However, the direction and magnitude of this force changes because of the orbital motion of the moon and also because of the motion of the bulge. So, to find the overall picture we have to take the average of the torque.

- **Averaging the tidal torque w.r.t. the motion of the moon and the bulge :**

  If the moon is located out of this paper or behind the back side of Fig. 3, then the tidal force on the right-side and left-side of the bulge is same in magnitude but oppositely directed. So, the resultant unbalanced tidal torque is zero. Therefore, for a complete moon's orbit, there are two points where the unbalanced tidal torque reaches a maximum value, and for other two points, it touches the minimum value. Therefore, the average tidal torque due to the moon's orbital motion is roughly $\langle \tau_{\rm tidal} \rangle = 0.5\, \tau_{\rm tidal}$.

  If we now consider the motion of the bulge, then again there are two points where $\tau_{\rm tidal}$ reaches maximum. Therefore, the average tidal torque is roughly

$$\langle \vec{\tau}_{\rm tidal} \rangle \sim \left(\frac{1}{2}\right)\left(\frac{1}{2}\right) \vec{R}_{\oplus} \times \vec{F}_{\rm tidal} \approx 0.25 \frac{GM_{\bullet}\, m_{\rm bulge} R_{\oplus}^2}{d^3} \cos\theta \sin\theta\ \hat{x} \tag{7}$$

- **Estimating precession rate :** The angular momentum of earth about its own axis $L = I\omega$, where $I \simeq (2/5) M_\oplus R_\oplus^2$ is the moment of inertia about its axis and $\omega = 2\pi/T$ is rotational frequency of the earth ($T = 24\,\mathrm{hrs}$). Therefore, the change in angular momentum $dL = L \sin\theta d\phi = \tau_{\rm tidal}\, dt$

$$\frac{d\phi}{dt} = \frac{\langle\tau_{\rm tidal}\rangle}{L\sin\theta} \approx \frac{5}{16\pi}\left(\frac{GM_\bullet}{d^3}\right)\left(\frac{m_{\rm bulge}}{M_\oplus}\right)\cos\theta\; T\,, \tag{8}$$

Note that, this is the tidal effect due to the moon. Following similar steps, we can estimate due to the sun (say, $\dot{\phi}_\odot$). Therefore, the total rate of change of angular momentum is

$$\dot{\phi}_{\rm total} = \dot{\phi}_\bullet + \dot{\phi}_\odot \approx \frac{5}{16\pi}\left(\frac{GM_\bullet}{d^3}\right)\left(\frac{m_{\rm bulge}}{M_\oplus}\right)\cos\theta\; T\left[1 + \left(\frac{M_\odot}{M_\bullet}\right)\left(\frac{d_\bullet}{d_\odot}\right)^3\right] \tag{9}$$

where $M_\bullet = 7.35\times10^{25}$ g, $m_{\rm bulge} = 0.0097\,M_\oplus$, $M_\odot = 2\times10^{33}$ g, and $d_\bullet = 3.85\times10^{10}$ cm, $d_\odot = 1.5\times10^{13}$ cm and $T = 24$ hrs. Putting all numerical values, finally we get $\dot{\phi}_{\rm total} \simeq 6.6\times10^{-12}$ rad s^{-1}. Therefore. the period of precession is $T_\phi = 2\pi/\dot{\phi}_{\rm total} \approx 30,200$ yrs which is close to the observed value $26,000$ yrs.

If the earth's rotation axis was not tilted or if the earth was not oblate:
If earth axis is not tilted, then bulge always at the same distance, so net unbalanced torque is zero, hence no axial precession. If earth was a perfect sphere (i.e., $R_{\rm p} = R_{\rm e}$) then effective bulge mass $m_{\rm bulge} = 0$ which again set the unbalance torque to zero.

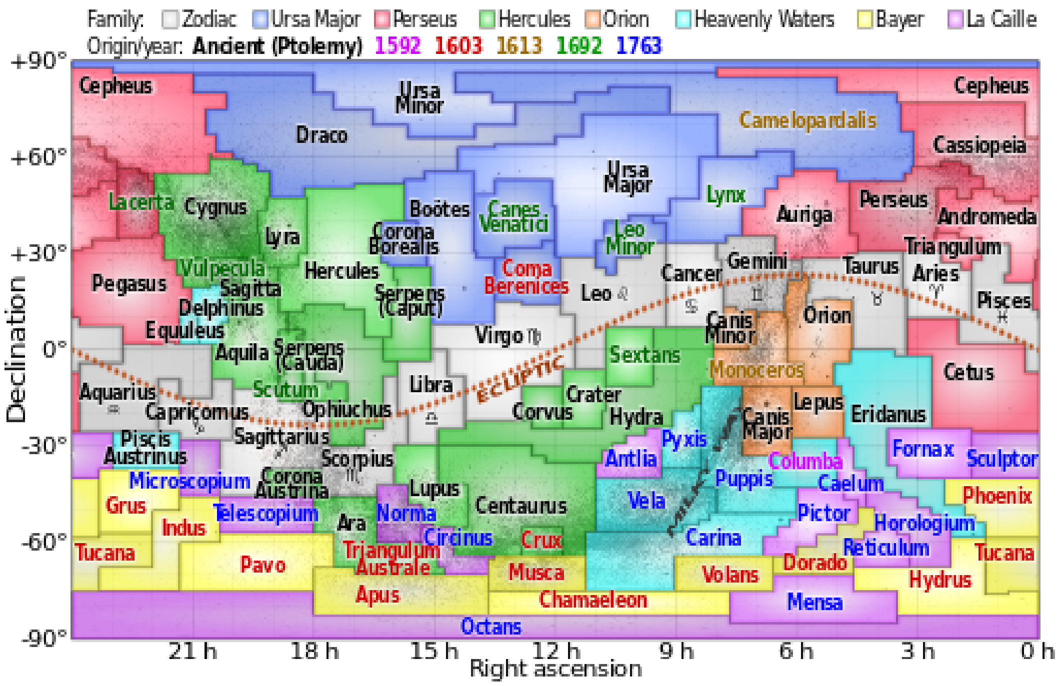

Figure 4: Right ascension (RA) and declination (DEC) of 88 constellations. The grey coloured region is called zodiac, a part of the sky within which the motion of our sun and all planets in our solar system are confined. The twelve constellations in the zodiac are the part of study in Astrology.

Age of Aries, Pisces, and Aquarius :
In $\sim$ 130 BC, Greek astronomer Hipparchus first identified that the vernal equinox is directed at the western extreme of the constellation Aries (known as the first point of Aries). So, they called that epoch as the age of Aries. From our discussion, we have

seen that the earth's axis precesses $\sim 1^o$ in 72 years, and therefore, the location of vernal equinox moves towards the western side as viewed from the earth (see Fig. 2). Currently, the vernal equinox is located in Pisces, slightly near the broader of Aquarius. We are in the transition regime between Pisces and Aquarius.

6. **Find out about the following iconic astronomical objects, their RA and Dec (where) known and describe them in two sentences. Comment about the nomenclature of these objects:**

(i) PSR 1913+16
A pulsar in a neutron star binary, it is the first binary pulsar discovered by Russell Alan Hulse and Joseph Hooton Taylor, Jr in 1974. It has a pulse period of 59 ms and an orbital period of 7.75 hours. The mass of each compact object is estimated to be $\sim 1.44 M_\odot$
RA: $19^h 13^m 12^s$
Dec: $16^o 01' 08''$

(ii) GRS 1915+105
A low-mass black hole binary having a mass $14 \pm 4_\odot$ at a distance $\sim$ 12.5kpc. It was discovered on Aug. 15 1992 by the WATCH all-sky monitor aboard GRANAT (hence the suffix GRS). It was the first such system discovered within the Galaxy.
RA: $19^h 15^m 11^s$
Dec: $10^o 56' 44''$

(iii) GW 150914
The first gravitational wave observed by the LIGO and Virgo collaborations-the result of a black hole merger. The waveform was detected on 14 Sept, 2015 (hence the name GW 150914). The initial black hole masses are $36M_\odot$ and $29M_\odot$ while the final black hole mass is $62M_\odot$ with $3.0M_\odot c^2$ radiated in gravitational waves.
(see: `http://journals.aps.org/prl/abstract/10.1103/PhysRevLett.116.061102`).

(iv) SN 1987A
A type II supernova in the LMC (a dwarf galaxy) about 51kpc away. It was the first supernova discovered in 1987 (hence the suffix A) more so during the era of modern telescopes.
RA: $05^h 35^m 08^s$
Dec: $-69^o 16' 12''$

(v) SN 1998bw
A type Ic gamma-ray burst supernova, the 76th supernova detected on 26 April, 1998 in ESO 184-G82 spiral galaxy. It was linked to GRB 980425 detected on April 25, 1998. It is the first supernova linked to a GRB.
RA: $19^h 35^m 03^s$
Dec: $-52^o 50' 46''$
Note:Supernova nomenclature follows a convention such that the prefix is SN, followed by the year of discovery and the alphabets A-Z for the first 26 discovered in that year and the subsequent ones are given the suffixes aa...zz for that year.

(vi) Messier 82
A star-burst galaxy (i.e one with unusually high star formation rate) at about 3.7Mpc in the constellation Ursa Major. It is the 82nd object catalogued by Charles Messier hence the name M 82

RA: $09^h55^m52^s$
Dec: $69^o40^{'}47^{''}$

(vii) 51 Peg b
An exoplanet about 15pc away in the constellation Pegasus. It was the first exoplanet discovered orbiting a main-sequence star (51 Pegasi, a G5V star). Its nomenclature followed the convention such that the prefix is the name of the host star followed by a lower-case letter starting with 'b' for planets orbiting the star.
RA: $22^h57^m28^s$
Dec: $20^o46^{'}08^{''}$

(viii) Sgr A*
The bright and compact radio source at the center of the Milky way. It is the location of the $4 \times 10^6 M_\odot$ black hole at the center of the Galaxy. It is in the constellation of Sagittarius at a distance of 8 kpc away.
RA: $17^h45^m40^s$
Dec: $-29^o00^{'}28^{''}$

(ix) 3C273
A quazar in the Virgo constellation, the first to be identified. It is one of the most luminous quazars known with an absolute magnitude -26.7 at a red-shift of 0.158. RA: $12^h29^m06^s$
Dec: $02^o03^{'}09^{''}$
Its nomenclature indicates that it is the 273rd object (by RA) in the third Cambridge catalogue of Radio sources (hence 3C) published in 1959.

(x)M 31
Otherwise called the Andromeda galaxy or NGC 224, It is the closest major galaxy to the Milky way at a distance of about 780 kpc away.In 1764, Charles Messier catalogued Andromeda as object M31.
RA: $00^h42^m44^s$
Dec: $41^o16^{'}09^{''}$

7. Find out the following velocities in km/s (i)the rotation velocity of the surface due to the earth's spin (ii)velocity of the earth around the sun (iii)velocity of the earth due to the moon's tug. The sun goes around the Galaxy at 220 km/s. Assuming spherical symmetry, estimate the galactic mass within the solar circle. Find out the earth's relative velocity with respect to the CMB rest frame. How is it measured? CMB rest frame correspond to the rest frame of the universe.

(i)At the equator, the earth's circumference ($2\pi R$) is $\sim 40,070km$ with a period of 24 hours. Thus;

$$v_{rot} = \frac{40070}{24} = 1670km/h \sim 0.5km/s$$

ii)Earth's orbit around the Sun covers on average $2\pi a$ where $a \sim 1AU$. Thus $v \sim 30km/s$
(iii)If they rotate about the common center of mass, then $m_1a_1 = m_2a_2$ and $a = a_1 + a_2$. From these equations, we have $a_2 = a(\frac{m_1}{m_1+m_2})$ and $m_1v_1 = m_2v_2$ where $v_1 = \omega a_1$ and

$v_2 = \omega a_2$. Now if m_1 is the mass of the earth with velocity v_1 and m_2 is the mass of the moon with velocity v_2, then taking $a \sim a_2$ we have;

$$v_1 = \frac{m_2 v_2}{m_1} \approx 0.01 km/s$$

where $v_2 = a_2 \frac{2\pi}{P}$

(iv)Balancing the centrifugal force on the Sun with the gravitational force of the Galaxy within the circle marked out by the Sun,

$$\frac{GMm}{a^2} = \frac{mv^2}{a}$$

so that;

$$M = \frac{v^2 a}{G} \approx 2 \times 10^{41} kg = 10^9 M_\odot$$

where M is the mass within the Sun-Galaxy center radius, m is the mass of the Sun and $a = 8kpc$

(v) The solar system is moving relative to the CMB at $368 \pm 2km/s$. This value is arrived at from the measurement of dipole anisotropy in the CMB as a result of Doppler shift caused by the motion of the solar system relative to the nearly isotropic blackbody radiation field (i.e the CMB)
(see: `https://ned.ipac.caltech.edu/level5/March05/Scott/Scott2.html`).

8. **The Sun contains majority of the mass of the solar system. What about angular momentum? The Sun's rotation period is about 25 days. Jupiter's mass is about $0.001 M_\odot$.**

The angular momentum of the Sun is

$$I\omega = \frac{2}{5} MR^2 \times \frac{2\pi}{P}$$

where P is the period of rotation of the Sun.

The angular momentum of Jupiter is ;

$$mrv$$

where m is the mass of Jupiter, r is the sun-jupiter distance and v is its orbital velocity around the Sun.

Comparing both values reveal that most of the angular momentum in the solar system is attributed to the planets with majority carried by Jupiter (see:
`http://www.haroldaspden.com/the-physics-of-creation/app 5.pdf`).