Fundamentals of Astrophysics, PH 217

Part I: Introduction, distances, coordinates, radiation, stellar types & binaries

1 Introduction

1. **Astrophysics:** *Applying physical laws* to understand observed astronomical phenomena (e.g., spectra of stars & statistical mechanics: higher ionization lines in massive/hotter stars; nuclear fusion in stellar interiors; all elements except H, He, Li are made in stars) and using astronomical data to *constrain/discover physical laws* at scales inaccessible in laboratories (e.g., neutrino oscillations; dark matter; dark energy).

2. **Relation to physics:** Almost *all branches of physics* used to understand some astronomical observation or other: compact objects (BHs, NSs, WDs) require knowledge of GR, nuclear & condensed-matter physics, quantum & statistical mechanics; accretion, outflows, winds & supernovae requires fluids & plasmas (statistical mechanics + electromagnetism); early universe requires high energy physics; ISM requires atomic physics & spectroscopy. Extreme range of temperatures (3 K CMB to 10^{12} K plasma around BH accretion disks) and densities (10^{-30} g cm^{-3} of the IGM to 10^{15} g cm^{-3} in NSs); CR protons up to 10^{20} eV compared to 10^{13} eV in LHC. *Can't do controlled experiments:* must infer from the meagre information from what the nature has set up. This means that robust physical arguments (even order of magnitude or back of the envelope estimate) preferred in building first models over detailed modeling (at least when observations are not sufficiently detailed; exception is, say the Solar surface, where observations are extremely detailed and sophisticated models are required to explain what is observed).

3. **Driven by observations:** *Multiwavelength* (radio, mm, IR, optical, UV, X-rays & $\gamma-$ ray) observations provide complementary physical constrains. Opening of new windows: gravitational waves & neutrinos (unaffected by the dense matter which absorbs photons). New observations driven by *technological advances* (e.g., interferometry over km scales in LIGO; adaptive optics enabling tracking of stars in Galactic center to put constrains on the mass of the SMBH). *Statistical arguments important* to make best use of limited information.

4. **Resources:**
   - *Primary text:* Astrophysics for Physicists (AP) by A. R. Choudhuri. *Others:* Astrophysics in a Nutshell (AN) by D. Maoz; The Physical Universe (PU) by F. H. Shu.
   - *Astrophysics papers & preprints:* ADS `http://adsabs.harvard.edu`, arXiv `http://arxiv.org/archive/astro-ph`, Google Scholar `https://scholar.google.co.in`.
   - *Astronomy databases:* NED `http://ned.ipac.caltech.edu`, CDS `http://cdsweb.u-strasbg.fr`.
   - *APOD:* `http://apod.nasa.gov/apod/astropix.html`.
   - *Instructor:* Prateek Sharma, D2-08, Department of Physics (prateek@physics).

2 Basic astrophysical units

Traditionally, CGS units are used in astrophysics and we will do the same in this course. However, in most cases it is more convenient to use astrophysical units such as the solar mass, Astronomical Unit (AU), etc.

1. **Mass:** Mass of astrophysical objects are expressed in units of solar mass $M_\odot = 2 \times 10^{33}$ g. Stellar masses lie in a range of 0.01 to 100 $M_\odot$. Jupiter mass is $0.001 M_\odot$ (2×10^{30} g) and Earth mass is $3 \times 10^{-6} M_\odot$ (6×10^{27} g). Total stellar mass of Milky Way is $\sim 10^{11} M_\odot$; total mass, which is dominated by dark matter, is $\sim 10^{12} M_\odot$. Galaxy clusters, the most massive gravitationally relaxed objects can go up to few times $10^{15} M_\odot$. An estimate of the mass of the visible universe: $\sim (4\pi/3)\langle\rho\rangle(c/H_0)^3 \approx (4\pi/3) \times 3 \times 10^{-30} \times (3 \times 10^{10}/2.3 \times 10^{-18})^3$ g $\sim 10^{22} M_\odot$ ($c = 3 \times 10^{10}$cm s^{-1} is speed of light in vacuum and $H_0 = 70$km s^{-1}Mpc^{-1} is Hubble constant).

2. **Length:**

- *Solar radius:* The radius of the Sun is $\approx 7 \times 10^{10}$ cm. Roughly 100 Suns can fit within the distance of the Earth from the Sun. White dwarfs have a radius similar to the Earth $\sim 10^{-2} R_\odot$; Neutron Stars are the size of a city ≈ 10 km; a solar mass BH has an event horizon of 3 km.
- *AU:* The average distance of the Earth from the Sun, called the *Astronomical Unit* (AU), is 1.5×10^{13} cm. This unit is useful for solar system studies. Neptune's orbit: 30 AU; Heliopause: 100 AU; Oort cloud: $10^3 - 10^5$ AU.
- *parsec (pc):* a short form for parallax of one arc second ($1'' = (1/3600)^0 = 1/3600 \times \pi/180 = 4.85 \times 10^{-6}$ radians), is a convenient unit to measure interstellar to cosmological distances. 1 pc is the distance at which 1 AU subtends and angle of $1''$. Therefore, 1 pc = 1 AU$/4.85 \times 10^{-6} \approx 3.086 \times 10^{18}$ cm ≈ 3.26 light years. The nearest star to us is Proxima Centauri at 1.31 pc. Galactic length scales are measures in kpc (1000 pc), distance between galaxies in Mpc (10^6 pc), and cosmological distances in Gpc (10^9 pc). The radius of the Milky Way's stellar disk is roughly 10 kpc; the disk height is ~ 100 pc. Hubble distance $d_H = c/H_0 \approx 1.4 \times 10^{28}$ cm ≈ 4 Gpc, approximately the radius of the observable universe with us at the center.

3. **Time:** Age of the Universe is ≈ 13.7 Gyr (1 Gyr = 10^9 yr). On the other extreme, pulsars emit pulses separated by as low as a millisecond. Year is used for long timescales and second for shorter timescales. Note that 1 yr $= 3.16 \times 10^7$ s $\approx \pi \times 10^7$ s. The age of the Sun is ≈ 5 Gyr. The Hubble time (of order the age of the Universe) is $1/H_0 \approx 14$ Gyr.

4. **Luminosity:** Solar luminosity $L_\odot = 4 \times 10^{33}$ erg s^{-1}. Known stars have luminosities in the range $10^{-5} - 10^6 L_\odot$. Stars are spherical objects emitting roughly *black body* radiation. *Luminosity* of an object is its inherent property independent of the distance of the observer, but the *flux* measured by the observer (energy crossing per unit area per unit time) goes as $L/(4\pi r^2)$ (the inverse square law; r is the distance of the observer from the source). Sun is so much brighter compared to other more luminous stars because it is so much closer to us.

   Magnitudes: Human eyes are sensitive to log of brightness, rather than brightness; i.e., objects of intensity I, $10I$, $100I$ appear to the eye as differing by the same value, rather than $100I$ appearing very bright. For this reason, astronomers traditionally use a logarithmic unit (called magnitudes) to quantify luminosity. *Apparent magnitude* is used to measure the flux and *absolute magnitude* is used for luminosity.

   Absolute bolometric (integrated over all frequencies: infrared, optical and UV) *magnitude* $\mathrm{M_{bol}} = 4.8 - 2.5 \log_{10}(L/L_\odot)$. Note that a more luminous star has a more negative absolute magnitude. Therefore, the Sun's absolute bolometric magnitude is 4.8. Absolute visual magnitude $\mathrm{M_V} = \mathrm{M_{bol}} - \mathrm{BC}$ (BC is *bolometric correction* that removes the luminosity contributions of wavebands other than "visual", about 0.5 microns). BC is a function of stellar temperature. Blue absolute magnitude $\mathrm{M_B}$ at blue band, 0.4 microns.

   Apparent bolometric magnitude $\mathrm{m_{bol}} = \mathrm{M_{bol}} + 5 \log_{10}(d/10 \text{ pc})$; i.e., the apparent magnitude matches absolute magnitude if the star is placed at 10 pc; it increases as we place the star farther away. Apparent visual magnitude $\mathrm{V} = \mathrm{M_V} + 5 \log_{10}(d/10 \text{ pc})$. Blue visual magnitude $\mathrm{B} = \mathrm{M_B} + 5 \log_{10}(d/10 \text{ pc})$. B$-$V color index, B-V=$\mathrm{M_B} - \mathrm{M_V}$, is a measure of the color of the star; i.e., the shape of the spectrum between V (0.5 micron) and B (0.5 micron) band. Very hot stars are blue and may have B$-$V = -0.3 and very cold stars have have B$-$V=1.5. Color index is a good indicator of the temperature of the stellar surface.

   §1.4 of AP gives a historical motivation for the definition of apparent magnitude.

   I am assuming that you all know about elementary facts about the motion of earth around the sun, and its relation to day/night and seasons, the reason behind eclipses and phases of moon and venus. Why are not all full moon nights lunar eclipses? etc. If you do not, please read it up from wikipedia. Here are some nice animations on the topic: `http://astro.unl.edu/animationsLinks.html`,`http://zingale.github.io/astro_animations/`.

3 Atmospheric transmission, space & earth based telescopes

Atmosphere allows radiation to pass only through certain windows, as shown in Fig. 1 (Figure credit: `http://gsp.humboldt.edu/olm_2015/Courses/GSP_216_Online/lesson2-1/atmosphere.html`). This is because of absorption due to molecules such as ozone, CO_2 and H_2O. Therefore, to observe gamma-rays, X-rays, IR from astronomical sources, one has to go to space. Radio observations can be done from the earth. Launching

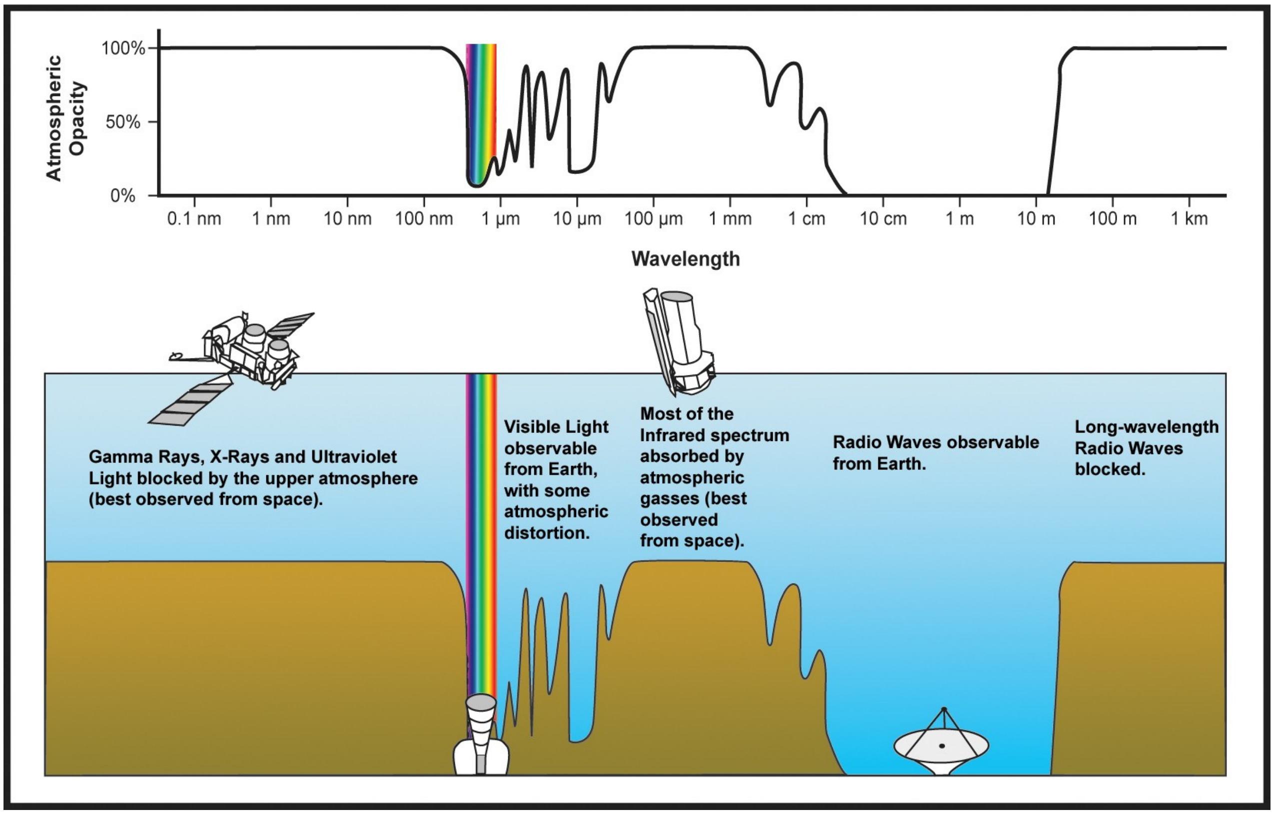

Figure 1: Only certain wavebands are accessible from ground. For others, we need to go above the atmosphere, into space.

satellites in space and managing them is more expensive. Some of the current space observatories are *Fermi* (gamma-rays), *Chandra* and *ASTROSAT* (X-rays), *Hubble* (NUV, optical and NIR), and *Spitzer* and *Herschel* (IR). Even in optical, atmosphere introduces "seeing" effects, and one needs to go to space to get the best spatial resolution (diffraction limit). Atmospheric turbulence and absorption are a function of location and altitude. That is why best astronomical observation sites are up on the mountains with still/cold air and low water vapor content.

1. **Diffraction limit:** Every optical system has a maximum resolution limit depending on the light-collecting primary aperture size. This diffraction limit is a consequence of wave nature of light (wave/physical optics, as opposed to ray/geometrical optics). Parallel light rays form a diffraction pattern with the first minimum occurring at an angle $\approx \lambda/D$ (D is the slit-size or the aperture diameter in case of an optical device). Even a point source will have this big an angular spread on imaging. Two point sources separated by less than and angle $\theta \approx \lambda/D$ will not be resolved by the instrument because their diffraction patterns (known as Airy pattern for circular apertures) are not sufficiently separated in angle. The exact calculation gives the diffraction limited angular resolution as $1.22\lambda/D$ (λ is the wavelength of observation and D is the diameter of the primary/objective mirror/lens). Space observations are not affected by atmospheric absorption and can, in principle, reach the diffraction limit.

2. **Seeing:** The biggest optical telescopes on earth have a 10 m diameter of the primary mirror. This gives a diffraction limit of $\approx 0.01'$. Turbulence in earth's atmosphere distorts the wavefront by introducing phase differences in different parts of the wavefront (turbulent refraction). This causes stars to twinkle and dramatically reduces the resolving power of the telescope. *Adaptive optics* is a technique which recovers the wavefront by moving the telescope mirrors accordingly to compensate for the phase differences in the wavefront. We need a star or an artificial source (created by laser) in the same part of the sky as the faint object that we are interested in imaging, to calibrate and get rid of atmospheric effects. *Speckle imaging*, in which very short exposure images are obtained, are least affected by temporal jittering. There are more related techniques for bright objects such as *lucky imaging*.

3. **Telescopes:** Telescopes are used to image far-away objects. The most important metric of a telescope is its aperture. A larger aperture gives a smaller diffraction limit and a bigger light collecting area. A higher angular resolution also means that faint objects stand out from the "sky," which refers to the faint radiation due to emission/scattering in the atmosphere. Magnification is not that important for astronomical telescopes but it can be easily controlled by reducing the eye-piece focal length. Most modern optical telescopes use reflecting parabolic mirrors, rather than refracting lenses for two reasons: (i) unlike mirrors, lenses are affected by aberrations throughout the bulk of the lens, and (ii) support structures for massive refractive elements also block the light therefore only the periphery of the lens is available for support structures (inc contract the full back of the mirror can be supported). Modern telescopes do not have a single mirror but are composed of several independent finely polished (irregularities finer than the wavelength) mirrors. X-ray telescopes such as *Chandra* use grazing incidence reflection to make images because the normally incident X-ray photons are simply absorbed. This makes their effective area much smaller than an optical telescope of the same primary mirror size.

   Modern optical telescopes have CCD (charged coupled device) cameras (similar to those in common digital cameras) at the focal plane to record the images. CCDs have a silicon slab divided into pixels. Photons falling on these pixels release photoelectrons and they are accumulated during the exposure. The charge recorded in each pixel during the 'readout' is proportional to the total number of photons reaching that pixel. This allows to make an image of the exposed object. Different detectors such as proportional counters, coded masks, particle detectors, etc. are used for detecting/imaging higher energy (X-ray and gamma-ray) photons.

   In radio band, typically an array of parabolic antennas receives a voltage signal as a function of time, which is recorded at the focal plane. The voltage signals are recorded and combined to make images. Voltage signals from spatially separated antennas can be combined to give an angular resolution of D corresponding to the separation between dishes, rather than the diameter of an individual dish. To achieve an even higher spatial resolution, the rotation of the earth can be used to synthesize an even finer 'beam', of order the size of the whole earth (this techniques is called VLBI, very large baseline interferometry). A key difference here, compared to higher frequencies is that both the phase and amplitude of the e.m. wave is recorded; therefore, signals contain more information. In contrast, at higher frequencies we typically count photons or measure intensities. Phase coherence is difficult to maintain at higher frequencies and interferometry is not possible. Radio waves are typically produced my non-thermal mechanisms; e.g., relativistic electrons gyrating in magnetic fields. In contrast, the spectrum of stars seen in UV, visible and IR is thermal black-body spectrum. The 2.73 K cosmic black body spectrum measured in microwaves & radio, is of course one of the best naturally occurring black body spectrum. For details on radio telescopes see: `https://public.nrao.edu/telescopes/radio-telescopes`.

   Sometimes the photons from the astronomical object are also dispersed by a prism or a grating to separate different frequency components of the light and to make a spectrum; i.e., the distribution of photons as a function of frequency. Such a device is called a spectrograph. Polarization of e.m. radiation emitted by an object (whether it is LP, CP, etc.) also carries important information about physical conditions, like the presence of magnetic fields and dust, scattering, etc. Some of the different modes of astronomical observations are: imaging, photometry (gathering photons), spectrometry (analyze frequency spectrum), polarimetry (polarization), astrometry (distances and velocities in the sky).

   See AP (chapter 1), AN (chapter 1) & PU (chapter 2) to learn more about telescopes.

4 Coordinate systems, proper motion, LOS velocity

We require two angles in the sky to locate an astronomical object in the sky; the distance of the object (the third coordinate) is more difficult to measure and is not used as frequently. Two important coordinates used in astronomy are celestial and Galactic (upper case 'G' stands for Milky Way or the Galaxy) coordinates.

1. **Celestial coordinates:** The most convenient coordinate system for earth based observers is the celestial (or equatorial) coordinate system. It is a simple extension of latitudes (declination or dec) and longitudes (right ascension or RA) used to indicate geographical locations (see the left panel of Fig. 2). The north celestial pole (where the earth's north pole pierces the celestial sphere; dec=90 deg) is close to the pole star. Distant stars are fixed in the celestial sphere but planets and the sun move relative to the fixed stars. The great circle on the celestial sphere vertically above the earth's equator is the celestial equator

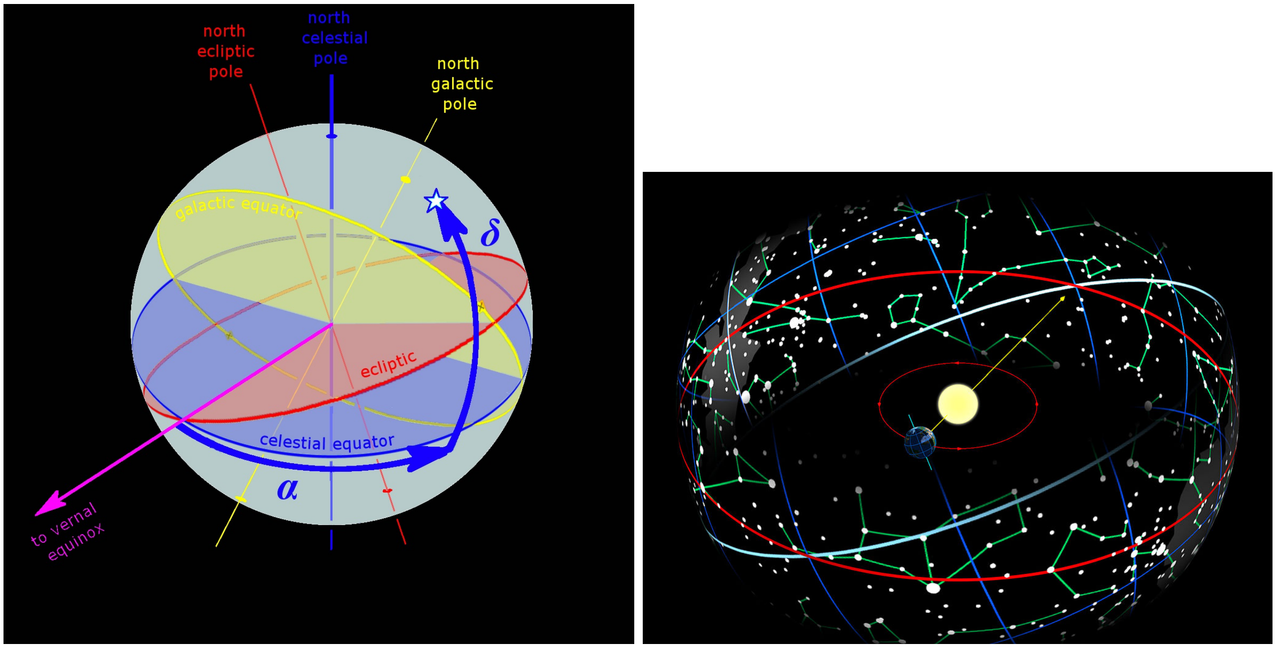

Figure 2: *Left panel:* Celestial coordinate system: RA=α; dec=δ are equivalent to longitudes and latitudes. Also shown are Galactic (the plane of MW) and ecliptic (the plane of the Sun and the planets) planes. The angle between celestial and ecliptic planes is 23.5 degrees. *Right panel:* The motion of the sun in the ecliptic plane against the constellations of fixed stars. Figure credit: wikipedia.

(dec=0 deg). Just as the longitude of Greenwich is chose to be 0 degree longitude, we need to choose a zero for RA. This is done with respect to the great circle called the ecliptic, the plane of the sun (and the solar system). The sun traces out a great circle in a year, called the ecliptic, with respect to fixed stars on the celestial sphere that is inclined at 23.5 degrees relative to the celestial equator (see the right panel of Fig. 2). This is nothing but the geocentric description of the motion of the tilted earth around the sun over a full year. The ecliptic cuts the celestial equator at two point, the vernal (spring) and autumnal equinoxes. The zero of RA is set to a point in the constellation Aries which corresponds to the vernal equinox. RA is typically expressed in hrs; 360 deg=24 hrs, and hence the celestial sphere rotates 15 deg in an hour about the celestial poles.

One subtlety with the celestial coordinates is that the earth's spin axis is precessing due to tidal torques with a period of about 26,000 years. This makes the location of equinoxes and NCP to precess at the same rate. Therefore the celestial equator and the zero of RA (coinciding with the vernal equinox) are defined with respect to a fixed time, say the year 2000, so that there is no ambiguity. Objects with large negative decs are not observable from the northern hemisphere. Only the objects that are sufficiently far away in RA from the sun are observable; otherwise they appear in the daytime and are overwhelmed by the sunlight. Equatorially mounted telescopes have one of their axes along the earth's spin axis and they can track stars by just adjusting along this axis.

2. **Galactic coordinates:** are useful for Galactic and extragalactic studies. The Galactic equator coincides with the disk of the Milky Way, and has a Galactic latitude b of zero deg. Solar system is still the center of the coordinate system. The Galactic center has the longitude l of zero deg. The stars and dust in the Milky way are confined to small bs.

Proper motion is the transverse (perpendicular to the line of sight) motion of stars relative to the fixed stars as measured from the location of the sun; it is typically measured in arcseconds per year. Stars with large proper motions are nearby. The motion of objects along the line of sight from the sun is called the **line of sight (LOS)** motion. LOS motion is measured using the *Doppler shift* of spectral lines. Note that we have to carefully account for the earth's motion around the sun and the sun around the Galaxy to interpret Doppler radial velocities. The Doppler technique was used to measure the tiny periodic wobble of the host star due to a planet; the first extrasolar planet around a main sequence star was discovered using this technique.

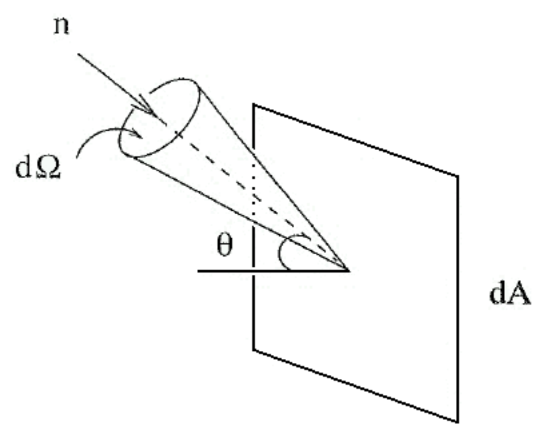

Figure 3: Specific intensity and radiative energy crossing per unit area along a solid angle.

Part II: Radiation & basics of stars

5 Black body radiation

The energy distribution of photons emitted by a body in which matter and radiation are in perfect thermal equilibrium is known as black body radiation. The energy density of black body radiation (measured in erg cm^{-3} Hz^{-1}) is given by

$$u_\nu = \frac{8\pi\nu^2}{c^3}\frac{h\nu}{e^{h\nu/kT}-1}, \tag{1}$$

where ν is the frequency, T is the temperature, h is Planck's constant, and c is the speed of light. This formula can be interpreted as follows. The energy of the photon is $h\nu$, $4\pi p^2 dp/h^3 = 4\pi(\nu^2/c^3)d\nu$ is the phase space volume density (in cm^{-3}; $p = h\nu/c$ is the momentum), there is a factor of 2 because photons can have a spin ± 1 (or two independent polarizations), and $(e^{h\nu/kT}-1)^{-1}$ is the Bose occupation number per mode. The black body radiation is isotropic and is completely described by Eq. 1. Planck's formula can also be understood from detailed balance argument of Einstein in which there are 3 microscopic processes in the interaction of matter and radiation: spontaneous emission, absorption and stimulated emission (recall Einstein's A & B coefficients).

Specific intensity: Radiation is characterized by the specific intensity (I_ν), the radiative energy crossing a surface per unit normal area per unit time per unit frequency per unit solid angle.

$$dE_\nu d\nu = I_\nu(\boldsymbol{r}, t, \hat{\boldsymbol{n}}) dA_\perp dt d\Omega d\nu,$$

where $dA_\perp = dA\cos\theta$, and θ is the angle between the surface normal and the direction of propagation of radiation. A complete description of the radiation field is provided by specifying I_ν everywhere at all times, and in all directions. Other quantities such as radiation flux and radiation pressure can be calculated by taking moments of I_ν w.r.t. θ. E.g., radiation flux at frequency ν is $F_\nu = \int I_\nu \cos\theta d\Omega$, and total (frequency integrated) radiation flux is $F = \int F_\nu d\nu$.

Since radiation traverses length cdt in time dt, the volume of the cylinder with the base dA is $cdtdA_\perp$. Therefore, radiation energy density per unit frequency is $u_\nu = dE_\nu/(cdtdA_\perp) = I_\nu d\Omega/c$, which is in general a function of the angle between the area and radiation. For isotropic radiation, such as black body, we get

$$I_\nu = \frac{c}{4\pi}u_\nu = \frac{2h\nu^3}{c^2}\frac{1}{e^{h\nu/kT}-1} \equiv B_\nu.$$

Radiation pressure is the flux of momentum perpendicular to a surface, $dE_\nu \cos\theta/(cdAdt) = I_\nu \cos^2\theta d\Omega/c$. Integrating over all angles, $P_\nu = (1/c)\int I_\nu \cos^2\theta d\Omega$. For isotropic radiation, $P_\nu = 4\pi I_\nu/3c$. Therefore $P_\nu = U_\nu/3$ for isotropic radiation.

From Planck's formula show that the energy density of BB radiation is $u = aT^4$, where $a = 8\pi^5 k^4/(15c^3h^3) = 7.6\times 10^{-15}$ erg cm^{-3} K^{-4} is the radiation constant. The flux is $caT^4/4 = \sigma T^4$, where $\sigma = 5.7\times 10^{-5}$ erg s^{-1} cm^{-2} K^{-4} is Stefan-Boltzmann constant.

From $B_\lambda d\lambda = B_\nu d\nu$, obtain B_λ. Two forms of Wien's law in frequency and wavelength. Rayleigh-Jeans regime and Wien tail. Give as HW problem.

6 Basic radiative transfer

Constancy of specific intensity along a ray in absence of matter: Consider a ray normally crossing two areas dA_1 and dA_2. The energy crossing dA_1 and dA_2 per unit time is $I_{\nu 1} dA_1 d\Omega_1 d\nu$, which by symmetry also equals $I_{\nu 2} dA_2 d\Omega_2 d\nu$. Equating the two, and noting that $d\Omega_1 = dA_2/R^2$ and $d\Omega_2 = dA_1/R^2$ (R is the separation between the two areas, we get $I_{\nu 1} = I_{\nu 2}$; i.e., specific intensity is constant along a ray path in vacuum. Mathematically, $dI_\nu/ds = 0$, where s is the coordinate along the ray path. We know that the flux decreases from a source as r^{-2}, with distance r from the source. But specific intensity includes division by solid angle which itself decreases by r^{-2}. Therefore, the ratio is a constant. Surface brightness, the ratio of flux and solid angle subtended by the object, is a related quantity that is an inherent property of the emitter (like luminosity) and does not depend on the distance of the observer. For a point source, the solid angle is limited by the diffraction limit and therefore the surface brightness falls with the distance. Similarly, at cosmological distances the luminosity and angular diameter distances are different and therefore surface brightness falls with distance. Moreover, absorption by intervening dust can reduce the surface brightness.

Basic radiative transfer along an absorbing column: The radiative transfer equation in presence of matter (which can emit and absorb) is

$$\frac{dI_\nu}{ds} = j_\nu - \alpha_\nu I_\nu, \tag{2}$$

where j_ν is volume emissivity (radiation energy emitted by matter per unit volume per unit time per unit solid angle) and α_ν is absorption coefficient.

If matter does not emit, but only absorbs, the solution of the radiative transfer equation is

$$I_\nu(s) = I_\nu(s_0) \exp\left[-\int_{s_0}^{s} \alpha_\nu(s')ds'\right].$$

Optical depth, general solution: A more useful coordinate for radiative transfer is the optical depth rater than s. Optical depth is defined as $d\tau_\nu = \alpha_\nu ds$, such that $\tau_\nu = \int_{s_0}^{s} \alpha_\nu(s')ds'$. Therefore, in absence of emission $I_\nu(\tau_\nu) = I_\nu(0)e^{-\tau_\nu}$. If $\tau_\nu \gg 1$, the object is known as optically thick (opaque). In the other limit $\tau_\nu \ll 1$, an object is optically thin (transparent). A source function is sometimes defines as $S_\nu = j_\nu/\alpha_\nu$. Therefore, the radiative transfer equation (Eq. 2) becomes

$$\frac{dI_\nu}{d\tau_\nu} = -I_\nu + S_\nu.$$

We can solve this equation by multiplying by an integrating factor e^{τ_ν}, $d(I_\nu e^{\tau_\nu})/d\tau_\nu = S_\nu e^{\tau_\nu}$. The solution of this equation is

$$I_\nu(\tau_\nu) = I_\nu(0)e^{-\tau_\nu} + \int_0^{\tau_\nu} e^{-(\tau_\nu - \tau'_\nu)} S_\nu(\tau'_\nu) d\tau'_\nu.$$

For a ray propagating through a uniform medium (i.e., S_ν is constant) this simplifies to

$$I_\nu(\tau_\nu) = I_\nu(0)e^{-\tau_\nu} + S_\nu(1 - e^{-\tau_\nu}).$$

In absence of a background radiation source ($I_\nu(0) = 0$), $I_\nu(\tau_\nu) = S_\nu(1 - e^{-\tau_\nu})$. In the optically thin limit ($\tau_\nu \ll 1$), $I_\nu(\tau_\nu) = S_\nu \tau_\nu = j_\nu L$ (L is the length of the ray path). I the optically thick limit ($\tau_\nu \gg 1$), $I_\nu = S_\nu$.

In this simple discussion, we do not consider scattering which is different from absorption and emission. Unlike microscopic absorption, in which photons is lost, scattering involves only a change in direction for the photon.

Kirchoff's law: For thermal emitters, $S_\nu = B_\nu(T)$, as BB spectrum is obtained in the optically thick limit. Therefore, $j_\nu = \alpha_\nu B_\nu(T)$, the famous Kirchoff's law, which roughly implies that a good thermal emitter is also a good absorber. Matter normally emits and absorbs in spectral lines, which are abundantly seen in optically thin nebulae. However, since stars are optically thick, we see a roughly BB spectrum (source function for thermal emission). Stars show absorption lines because density, temperature are not uniform and cooler overlying material can absorb the background BB.

Opacity, photon mean free path: The probability of a photon traveling to an optical depth of τ_ν is $e^{-\tau_\nu}$. Therefore, the average optical depth traversed by a photon is $\int_0^\infty \tau_\nu e^{-\tau_\nu} d\tau_\nu = 1$. The mean physical distance traveled by a photon (or the photon mean free path) is, thus, $\lambda_{\rm ph} = 1/\alpha_\nu = 1/(n\sigma)$, where σ is the absorption cross-section of photons at frequency ν and n is the number density of absorbers. Sometimes, opacity $\kappa \equiv n\sigma/\rho = \sigma/(\mu m_p)$ (unit cm^2 g^{-1}; characterizes absorption cross-section per unit mass) is used to characterize absorption properties of a medium.

Chapter 1 of Rybicki and Lightman's book Radiative Processes in Astrophysics is a good introduction to radiative transfer.

Local thermal equilibrium: The concept of local thermal equilibrium is very important. We can consider equilibria of different kinds: collisional, radiative, ionization/recombination, chemical, nuclear; etc. Collisional equilibrium involves collisions between molecules (Coulomb collisions for a plasma) that enforce the Maxwell-Boltzmann distribution of velocities of molecules. Similarly, one can consider matter-radiation equilibrium between absorbing/emitting atoms/molecules and photons. The microscopic interaction in this case is emission (spontaneous and stimulated) and absorption of photons by matter. These interactions are necessary to establish the Planck spectrum of a black body. A Maxwellian velocity distribution is obtained if the collisional mean free path is much smaller than the system size; or equivalently, the collision time is much shorter than the timescales of interest. Similarly a BB spectrum is obtained if the photon mean free path is much shorter than the system size. The temperature of the radiation in earth's atmosphere is close to a BB, but with the temperature of the solar photosphere (6000 K) but the air is just 300 K. Therefore, atmosphere is in collisional equilibrium (i.e., gas molecules have Maxwell-Boltzmann velocity distribution) but matter and radiation are not in equilibrium (and hence the spectrum is not a black body at 300 K but at the temperature of the solar photosphere). The solar surface, on the other hand, is in both collisional and radiation equilibrium and hence is a good BB.

Saha equation: Another instance of equilibrium is between collisional ionization and recombination ($H \leftrightarrow H^+e^-$). Applying statistical mechanics to ionization/recombination of hydrogen, on can obtain the equation for ionization fraction as a function of density and temperature of a gas in equilibrium, namely (**Saha equation**),

$$\frac{n_i n_e}{n_H} = \frac{(2\pi m_e k_B T)^{3/2}}{h^3} e^{-\chi/k_B T}. \tag{3}$$

Here symbols have their usual meanings (n_e: electron number density; n_i: proton number density; n_H: neutral hydrogen number density; χ: 13.6 eV, ionization potential of Hydrogen). It is instructive to plot $x = n_e/(n_e + n_H)$ as a function of temperature for different number densities ($n = n_e + n_H$). It is interesting to note that for typical astrophysical densities, the ionization fraction is 0.5 at $\sim$ 5000 K, much smaller than the temperature corresponding to the ionization potential (1.4×10^5 K). This somewhat surprising result is because of the large degeneracy of a free electron, which typically has much larger available volume compared to its de Broglie wavelength. Once H is ionized, suddenly the available phase space volume increases, favoring the ionization reaction even at temperature smaller compared to χ.

Brightness temperature, color temperature and effective temperature: There are some popular ways to assign "temperature" to astronomical objects. **Brightness temperature** is defined as $B_\nu(T_b) = I_\nu$. In the Rayleigh-Jeans part of BB spectrum $h\nu/k_BT \ll 1$, $T_b = c^2 I_\nu/(2k_B\nu^2)$. For a BB the brightness temperature equals the true temperature. In general, T_b is unrelated to the temperature of the medium (T). In low frequency limit, the radiative transfer equation can be written in terms of temperatures, $dT_b/d\tau_\nu = -T_b + T$. For a constant temperature medium, $T_b = T_b(0)e^{-\tau_\nu} + T(1 - e^{-\tau_\nu})$. **Color temperature** of an emitter is essentially obtained by applying Wien's displacement law for the measured I_ν. **Effective temperature** relates the flux emitted from a body, such that $\sigma T_{\rm eff}^4 \equiv F$. For BBs, all these definitions equal the temperature of the photon-matter in equilibrium.

Karmer's & Thomson opacity: Opacity, or the resistance of matter to the passage of photons, depends on the material properties which depend on density and temperature (and composition) of matter. For temperatures between 10^4 K and 10^7 K, matter can absorb (and emit, remember Kirchoff's law) due to bound-bound (electron is bound to the atom both before and after the absorption of photons; more important at lower temperatures at which the gas is not fully ionized), bound-free (electron is bound before absorption and freely moving after the transition) and free-free (freely moving electron both before and after absorption of photons) transitions. The frequency-integrated opacity for thermal gases in $10^4 - 10^7$ K is often approximated as $\kappa \propto \rho/T^{3.5}$. At $T < 10^4$, hydrogen recombines and photons do not have enough energy to excite lines and hence α (absorption coefficient) should decrease below this temperature. Similarly, at higher temperatures $T > 10^7$ K, the plasma is fully ionized and free-electron scattering (Thomson scattering) opacity dominates.

Free electrons oscillate in presence of radiation and emit e.m. waves in directions away from the incident direction. This is elastic scattering in the limit $h\nu \ll m_e c^2$, is known as Thomson scattering. Thomson scattering cross-section is $\sigma_T = (8\pi/3)r_e^2$, where $r_e = e^2/(m_e c^2)$ is the classical radius of electron. This is a dominant scattering mechanism for fully ionized plasmas. The Thomson scattering cross section is $\sigma_T = 6.6 \times 10^{-25}$ cm^2

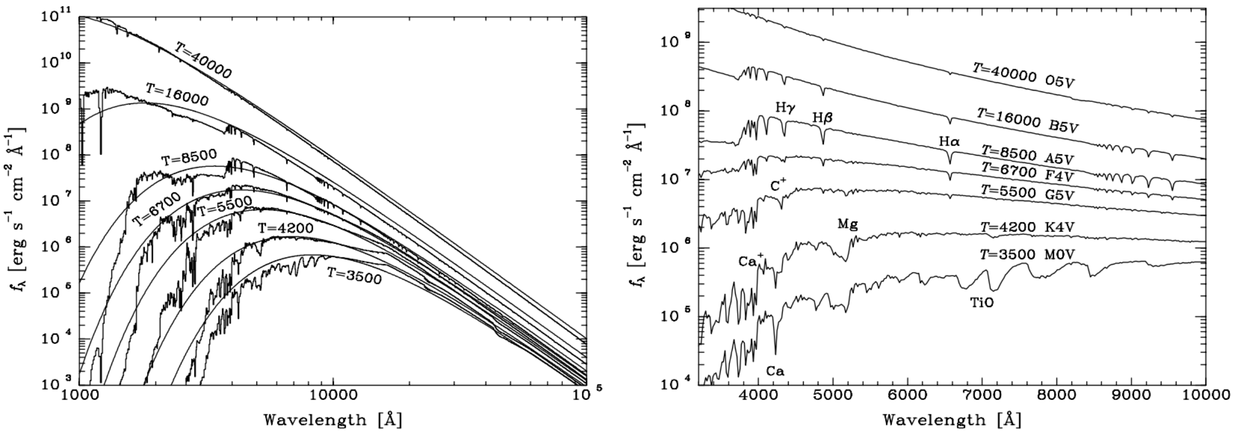

Figure 4: Model spectra for different spectral types of stars adapted from AN. *Left panel:* shows the underlying black-body spectrum (notice λ^{-4} in the long wavelength portion of the spectrum, as expected in the Rayleigh-Jeans part); *Right panel:* focuses on absorption lines.

and Thomson opacity is $\kappa_T = n_e \sigma_T / \rho = 0.4$ cm^{2}g^{-1} for hydrogen plasma. Thomson scattering is not relevant for stars as they are cooler and Kramer's opacity is more relevant. In the early universe, before the CMB era, the universe was fully ionized and photons were scattered by free electrons in the plasma. As plasma cooled and hydrogen recombined, all the free electrons became bound and the universe became transparent to photons.

Emission and absorption mechanisms in thermal gas (e.g., in form of opacity) is relevant for quantitative stellar structure calculations, and for other problems involving matter radiation interaction. Opacity tables are provided by some well-known databases.

Photon diffusion timescale from the center of the sun: It is useful to estimate the photon diffusion timescale from the center of the sun. The Kramer's opacity relevant for central temperature and density of the sun is ~ 1 cm^2 g^{-1}. Thus, the average mean free path is $\sim 1/(\rho\kappa)$. Taking average density $\rho \sim 1$ g cm^{-3}, the mean free path is ~ 1 cm. Here we are trying to make a very crude order of magnitude estimate and do not consider variations as a function of the solar radius. The solar radius is $\approx 7 \times 10^{10}$ cm, much larger than the photon mfp. Thus, photons diffuse out from the center. Also, in reality photons are getting absorbed and re-emitted, typically at lower frequencies, as they travel out. But we do not take these complications into account.

The displacement from the center of the sun $\boldsymbol{l} = \boldsymbol{l_1} + \boldsymbol{l_2} + ... + \boldsymbol{l_N}$ after N steps of the random walk. The average displacement after N steps of the random walk is $\langle \boldsymbol{l} \rangle = 0$, as photons have equal probability of going in all directions. The mean square displacement is $\langle l^2 \rangle = \langle l_1^2 \rangle + \langle l_2^2 \rangle + ... + \langle l_N^2 \rangle = N\lambda_{\rm mfp}^2$. The cross terms vanish because the different steps of the random walk are independent. Thus the rms displacement from the center is $N^{1/2}\lambda_{\rm mfp}$. Equating this to the solar radius ($R_\odot = 7 \times 10^{10}$ cm), we obtain $N \sim (R_\odot/\lambda_{\rm mfp})^2 \sim 5 \times 10^{21}$ steps of the random walk are required for a typical photon to diffuse out of the center. The time is $\sim N\lambda_{\rm mfp}/c = R_\odot^2/(\lambda_{\rm mfp} c) \sim 5.2 \times 10^3$ yr (a more careful estimate gives few $\times 10^4$ yr; the photon diffusion coefficient, from random-walk arguments, is $\lambda_{\rm mfp} c$), much longer than the solar light crossing time $R_\odot/c$. The photon diffusion coefficient, based on random walk arguments is $\lambda_{\rm mfp} c$. This means that if the sun were to stop nuclear burning at the center today, the output luminosity will only be affected after 10^4 yr!

7 Stellar types & temperatures

Stars are usually classified into spectral types, which in the order of decreasing temperature are O, B, A, F, G, K, M (a common mnemonic used to remember this is "O Be A Fine Gal/Guy, Kiss Me"). The hottest most luminous (and also the hottest; physical characteristics of stars such as mass, temperature and luminosity have a one-to-one relationship) stars are O stars and the coolest stars are M stars (two more types L and T have been added to stellar types; these are brown dwarfs, intermediate between stars and giant planets). There are subtypes within a given spectral type between 0 and 9, with a larger number indicating a lower temperature. Our sun is a very ordinary G2 star with $T_{\rm eff} = 5800$ K.

Fig. 4 shows the model spectra for stars with different (effective) temperatures. One typically sees absorption lines in stellar spectra because the solar atmosphere is cooler than the photosphere, and the atoms and molecules in the atmosphere resonantly scatter (an atom absorbs in a resonant lines but re-emits in some other direction) or absorb (photon is absorbed by exciting an electron from a lower to a higher level but relaxation happens non-radiatively via collisions) the background BB radiation. Chromosphere (with 10^4 K lies on top of the photosphere) and corona (extended, hot 10^6 K atmosphere) can produce emission lines (as they are hot and dilute). The variation in absorption lines in different spectral types happens because of temperature and not because of metallicity (in astronomy all elements except for H, He are called metals). Most stars have very similar compositions, containing 75% by mass of H and 23% by mass of He, and only 2% in metals.

Hydrogen lines: Since H is the most abundant and the simplest atom, it is useful to recall the spectral lines of H. Recall the energy of the nth bound energy level is 13.6 eV. Its most useful to express the transition wavelength as a function of n_1 and n_2, the transition energy levels,

$$\lambda_{12} = \frac{hc}{E_{12}} = \frac{911.5 \text{ Å}}{1/n_1^2 - 1/n_2^2}. \tag{4}$$

Lyman series: corresponds to a transition to $n = 1$ ground state. 1216 Å Lyα (n =2 to 1), 1026 Å Lyβ ($n = 3$ to 1), 973 Å Lyγ (n =4 to 2), ..., Ly continuum ($n = \infty$ to 1) 912 Å. Lyman lines appear in UV.
Balmer series: corresponds to a transition to $n = 2$ state. 6570 Å Hα (n =2 to 1), 4867 Å Hβ ($n = 3$ to 1), 4345 Å Hγ (n =4 to 2), ..., Balmer continuum ($n = \infty$ to 2) 3650 Å. Balmer lines appear in optical.
Similarly, there are Paschen ($n = 3$), Brackett ($n = 4$) and Pfund ($n = 5$) transitions at longer wavelengths. Balmer lines are particularly important as they can be observed from ground at optical wavelengths.

Now, coming back to the absorption lines in the right panel of Fig. 4, the Balmer absorption lines are very prominent for the A stars with the effective temperature of 8500 K. This temperature is the sweet spot for the presence of hydrogen lines because this is hot enough to have enough H atoms in excited ($n = 2,\ 3$,...) levels and not too hot to completely ionize H (this is what happens for O stars; see Saha equation; Eq. 3). Lower mass (and correspondingly temperature) stars do not have enough H atoms in excited ($n = 2,\ 3$,...) levels and they show absorption lines due to Ca^+, Mg, Na, etc. and even molecular absorption such as TiO bands (seen in 6600 to 8800 Å) shown in Fig. 4 for an M star. The stars are typically classified according to the prominence of different absorption line features. A and F stars have prominent Balmer lines and a sharp drop at Balmer continuum (3650 Å). O and B stars have absorption lines of higher ionization species such as He and He^+, which can only be excited at higher temperatures. Of course, O stars being hot are bluer as compared to the cooler stars (just a consequence of Wien's law for BBs). While molecules and metals have low abundances, they can have stronger absorption at appropriate temperatures because of a higher line opacity.

8 Binary systems

Binary stars are very common and they allow dynamical mass measurement of stars in some cases. Lessons learned from binary star observations are applied to the young field of exoplanet (planets beyond our solar system) detection. There are various classes of binary systems. **Visual binaries** are the ones where the two (or more) members of the binary are visually resolved. In **astrometric binaries** one can observe the minute motion of one of the stars due to the presence of a companion, even though the companion is too faint to be seen. In **eclipsing binaries**, one observes two eclipses every time period in the light curves (flux as a function of time); this is only possible if the binary is sufficiently inclined to our line of sight. The depths and durations of the eclipses give us information about mass and radius of the two stars. A **spectroscopic binary** is spatially unresolved but its spectrum shows blue and red shifted absorption lines oscillating at the time period of the binary. The frequency/wavelength shift gives velocity via the Doppler formula, $\Delta\lambda/\lambda = v/c$. Sometimes, only one of the members have strong enough lines but the periodic variation in red and blue shifts tell us that it is a binary system (this is the idea behind Doppler method for searching for exoplanets).

The two stars in a binary system go around the common center of mass, and a circular orbit is generally a good assumption. (Can you make a guess, why? Think how planets/stars are born? When do you think this assumption may break down?) A simple application of circular orbits in a 2-body gravitational problem gives us a handle on the binary parameters (masses, separations, etc.). This is covered in AN and in HW2. Please read these carefully.

Here I illustrate the **Doppler technique** used for the detection of extrasolar planets. The idea is to infer the presence of a planet by the slight (10s of m s^{-1}, corresponding to $\Delta\lambda/\lambda \sim 10^{-7}$!) and periodic variation in spectral absorption lines of the host star. Let us consider masses M_1 and M_2 orbiting around each other in a circular orbit about the common center of mass. From Kepler's law $G(M_1 + M_2)/(r_1 + r_2)^3 = \Omega^2$, where Ω is the angular frequency ($2\pi/T$ where T is the time period is measured from the periodic shift of spectral lines),

r_1 and r_2 are the radii of circular orbits about the common center of mass. The line of sight velocity of the star $v_{1,\mathrm{los}} = v_1 \sin i$, where i is the inclination of the orbit; $i = 0$ for a face-on orbit (of course no Doppler shift is observed in this case) and $i = 90^0$ for edge-on. $r_1/r_2 = v_1/v_2 = M_2/M_1$, as both members of the binary are orbiting at the same frequency preserving the center of mass location. Using this, the Kepler's law becomes

$$(M_1 + M_2)\sin^3 i = \frac{T(v_{1,\mathrm{los}} + v_{2,\mathrm{los}})^3}{2\pi G}.$$

If we have Doppler measurements from both stellar components (i.e., $v_{1,\mathrm{los}}$ and $v_{2,\mathrm{los}}$ are known), there are three unknowns in this equation, M_1, M_2 and i. For exoplanets, $v_{2,\mathrm{los}}$ is not known but we can use $v_2 = (M_1/M_2)v_1$. For planetary companion, $M_2 \ll M_1$, and the above equation becomes

$$M_2 \sin i \approx \left(\frac{T}{2\pi G}\right)^{1/3} v_{1,\mathrm{los}} M_1^{2/3}.$$

Here again, we have 3 unknowns and one equation. From measuring the stellar spectrum (the stellar structure theory is very well developed) we can accurately get M_1. Therefore, Doppler measurement only gives $M_2 \sin i$; M_2 can be much greater than $M_2 \sin i$ if the orbit is close to face on. Sometimes, we detect planetary transits (i.e., eclipses due to planets; read about the Kepler satellite which has detected 1000s of planetary candidates using the transit technique), and in those cases i is contained to be close to 90^0 and $M_2 \sin i$ is close to M_2.

In detail, exoplanet observers have to worry about complications such as elliptical orbits, influence of other planetary/stellar companions, etc. Also, very importantly, one needs to subtract earth's rotation and motion around the sun very carefully in order to not detect all exoplanets with a 24 hour or 365.25 day orbits!