Fundamentals of Astrophysics, PH 217

Part II: Stars, their evolution, and end states

1 Introduction

Stars are the 'hero' of astrophysics (after all it is the physics of stars). A spherical, hydrostatic approximation is good for most stars, and therefore stars are simpler to model than say disks, which are essentially multi-dimensional. Key physical aspects of stars are very well understood. Elements from thermodynamics and statistical physics, nuclear physics, radiative transfer, and fluid dynamics are used to understand stars. Stars, emitting profusely in optical, are also the objects observed for the longest time.

Sun as a star: Sun is the only star that is spatially resolved; all others are point sources as observed from earth. Because of its proximity we know a great deal about our sun. Nuclear fusion happens only in the center because the temperature is high enough only in the core ($\lesssim 0.2 R_\odot$). The energy produced at the center is transported radiatively via photons diffusing out of the center. Beyond $0.7 R_\odot$, radiative energy transport is not efficient in transporting the energy outwards and convection (similar to the familiar bubbling of water) takes over. Granulation observed in the solar photosphere directly shows hot fluid rising and cooler regions sinking. One of the early (in 1940s) puzzles was regarding the *source of opacity close to the solar photosphere*, where the temperature is 6000 K, too low to have H (most of which is in ground state) absorbing photons (excitation of H from $n = 1$ to $n = 2$ requires 10.2 eV). It was realized that H^-, a loosely bound structure with an ionization potential of 0.75 eV is the main opacity source near the solar photosphere.

2 Basic equations

Stars are hydrostatic balls with the higher pressure toward the center supporting the gravitational pull on the shells. The equation of hydrostatic equilibrium is $dp/dr = -\rho g = -GM(r)\rho/r^2$. The enclosed mass within radius r is related to the mass density as $dM/dr = 4\pi r^2 \rho$. These are two equations for three unknowns (ρ, M and p). We need an additional equation, which typically is the energy transport equation. Sometimes (especially before the energy generation mechanism in the stars was known) polytropic equations of state are used which relate the pressure and density as $p = \rho^\Gamma$, where Γ is the polytropic index. Such an approximation is not valid in general, but alright in some cases as white dwarfs and fully convective stars. This relation makes the number of equations equal to the number of unknowns.

Virial theorem: Multiplying the HSE equation by $4\pi r^3$ and integrating over the whole star, gives

$$\int_0^{R_\star} 4\pi r^3 \frac{dp}{dr} dr = -\int_0^{R_\star} \frac{GM(r)\rho}{r^2} 4\pi r^3 dr.$$

The LHS when integrated by parts, and by noting that $p(R_\star) = 0$, gives $-\int_0^{R_\star} 12\pi r^2 p dr = -2E_T$, where E_T is the total thermal energy of the star. Here we have assumed that $u = 3p/2$, valid for a non-relativistic ideal gas. The RHS is $\int_0^{R_\star} \rho\phi 4\pi r^2 dr = E_G$, the gravitational energy of the star. Therefore, HSE implies, $E_T + E_G/2 = 0$, a result known as the viral theorem for stars. The same is true for a collection of particles in a gravitational potential. This implies that the total energy $E = -E_T = E_G/2 < 0$; i.e., the star is bound. Initially unbound matter collapses to form a star with a total energy which is negative ($E_G/2$); remaining positive energy $-E_G/2$ is available to be radiated. This is the principle behind Kelvin-Helmholtz contraction. In fact, the fraction of energy lost via radiation in Kelvin-Helmholz contraction is equal to the kinetic energy retained by the star, the sum of which is equal and opposite to the gravitational binding energy of the star (as required by total energy conservation). Note that if one does the same calculation for a relativistic gas (for which $u = 3p$; applicable for radiation dominated stars), the total energy is zero and the star is barely bound. Unlike normal stars, such a star is unstable. These considerations have profound implications for a star's stability.

The viral theorem can be used to express the volume averaged pressure inside the star as $\langle p \rangle = -E_G/(3V)$ ($V = 4\pi R_\star^3/3$ is stellar volume). The volume averaged pressure is $\sim GM_\star^2/(3VR_\star) \approx 10^{15}$ erg cm^{-3} for the sun (here we have estimated the average gravitational energy as $-GM_\star^2/R_\star$). The *virial temperature* is defined as $(3/2)nk_BT_v = GM_\star^2/(2R_\odot)$, or $T_v = GM_\odot m_p/(6k_BR_\odot) = 0.4$ keV (here we assume that the number density the $2\rho/m_p$ for a H plasma). This is an order of magnitude estimate for the temperature in the core of the sun.

Energy transport via radiation: As already mentioned, nuclear energy is generated in the cores of stars and transported radiatively till $0.7 R_\odot$, beyond which energy transport happens via convection. The energy

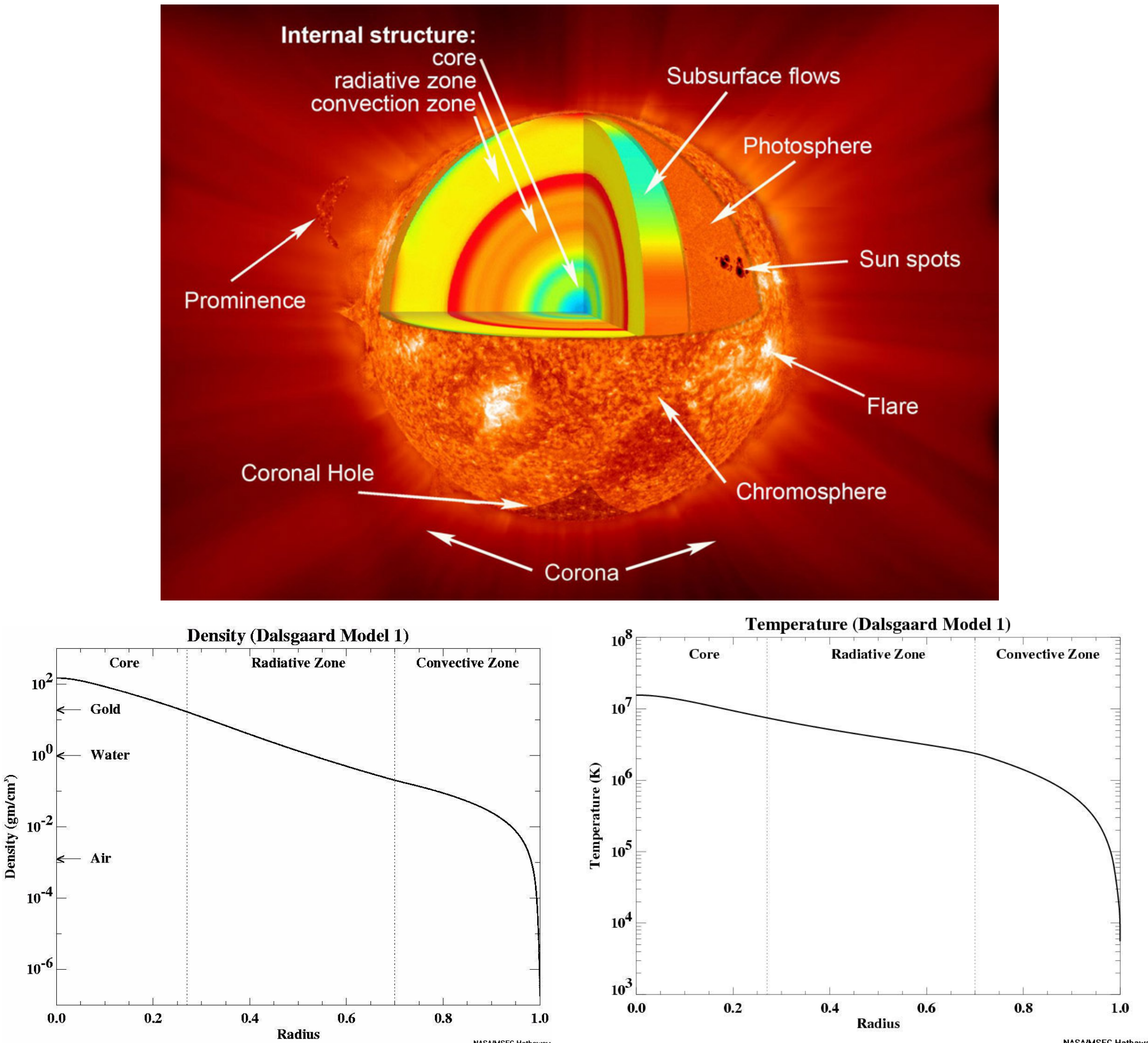

Figure 1: Different parts of the sun. Top panel is a visual representation, while the bottom panels are density and temperature profiles from the solar models. Inside the radiative zone entropy (or p/ρ^γ; γ is the adiabatic index) increases with radius and within the convection zone it is almost flat. Outside the photosphere there is chromosphere ($T \sim 10^4$ K), which emits strong Hα *emission* lines. Beyond this is the solar corona with a temperature of 10^6 K. Images are taken from the web.

transport equation is given by $dL_r = 4\pi r^2 dr \rho \epsilon$, where L_r is the radial luminosity (energy flowing per unit time across radius r) and ϵ is the energy production rate per unit mass due to nuclear fusion. In case energy is transported via radiation, the radiative energy flux ($= L_r/[4\pi r^2]$) is given by $-Ddu/dr$, where $D = \lambda_{\rm mfp} c/3$ (factor of 1/3 comes because photon motion in θ and ϕ does not contribute to radial diffusion) is the radial diffusion coefficient for the photons and $u = aT^4$ is the radiation energy density. Here it is assumed that the photon mean free path is tiny compared to the scales of interest. The radial luminosity flux is given (in terms of opacity) by

$$L_r = -4\pi r^2 \left(\frac{c}{\rho\kappa}\right) \frac{d}{dr}\left(\frac{aT^4}{3}\right).$$

Energy is transported outward from higher to lower temperatures because of photon diffusion.

Energy transport via convection: Under some conditions, for example when opacity is large or temperature is low, energy transport can take place via bulk fluid motions. In convection hot blobs rise and cooler ones sink, transporting energy from inner to outer regions. Convection is observed in boiling water where one easily sees overturning fluid motion.

In class we analyzed the stability of a fluid blob displaced in an atmosphere under hydrostatic equilibrium. In the limit the the blob is displaced slowly (compared to the sound crossing time), the blob is always in pressure balance with the background atmosphere. Also, in the adiabatic limit the blob maintains its entropy or p/ρ^γ (γ is the adiabatic index). These considerations give that the displaced blob will be heavier than its surroundings if the background atmosphere's entropy increases with radius, or in other words, if

$$\frac{d}{dr}\ln\left(\frac{p}{\rho^\gamma}\right) > 0.$$

This condition can be written in terms of pressure and temperature profiles as

$$\left|\frac{dT}{dr}\right| < \left(1 - \frac{1}{\gamma}\right)\frac{T}{p}\left|\frac{dp}{dr}\right|.$$

Of course in stars dT/dr and dp/dr both are negative. While dp/dr is always negative in HSE, temperature gradient can have any sign depending on heating and cooling. E.g., the temperature increases with height in the stratosphere. Typically the luminosity of the star can be carried by very subsonic motions, and once convection ensues, it is so effective that the temperature gradient is pinned almost at the marginal value; i.e.,

$$\frac{dT}{dr} \approx \left(1 - \frac{1}{\gamma}\right)\frac{T}{p}\frac{dp}{dr}$$

in the convective zone.

3 Constructing stellar models

Collecting all the stellar structure equations together, we have

$$\frac{dM}{dr} = 4\pi r^2 \rho, \quad (1)$$

$$\frac{dp}{dr} = -\frac{GM(r)\rho}{r^2}, \quad (2)$$

$$\frac{dL}{dr} = 4\pi r^2 \rho \epsilon, \quad (3)$$

$$\frac{dT}{dr} = \frac{-3}{4ac}\frac{\kappa\rho}{T^3}\frac{L}{r\pi r^2}, \text{ in the radiative zone} \quad (4)$$

$$\frac{dT}{dr} = \left(1 - \frac{1}{\gamma}\right)\frac{T}{p}\frac{dp}{dr}, \text{ in the convective zone.} \quad (5)$$

Here there are 6 unknown variables are $M, \ \rho, \ p, \ L, \& \ T$. We assume an equation of state $p = p(\rho, T, X_i)$ (X_i denotes composition) and a form for $\epsilon(\rho, T, X_i)$ and $\kappa(\rho, T, X_i)$. These are the three constitutive equations which depend on the microscopic properties of the matter and are assumed to be known. Thus we have an equal number of equations and unknowns. The natural boundary conditions for our four ODEs are $M(r=0) = 0$, $L(r=0) = 0$, $p(r = R_\star) = 0$ and $M(r = R_\star) = M_\star$.

Now the question is which equation (Eq. 4 or 5) should we use for energy transport. Typically we assume radiation transport and check if the profiles so obtained are convectively unstable. If they are, we assume convective energy transport (Eq. 5) in those regions and re-integrate the stellar model. Another complication is that the boundary conditions are specified both at $r = 0$ and $r = R_\star$. The way to solve these ODEs is to assume a central density and temperature and integrate them out till the radius where p and ρ become 0. One needs to play around with these central density/temperature guesses till we obtain the same value of the radius at which both p and ρ vanish. The actual calculation of stellar models is somewhat complicated but we have touched upon the essential ideas.

Abundances and relation between p & ρ: By comparing the kinetic energy of stellar plasma and the mean potential energy, it is easy to show that the plasma behaves like an ideal gas (see Q1 in HW3) with $p = nk_BT$, where n is the total number density of particles. We want to relate the number density (n) and the mass density (ρ). Usually the stellar abundances are specified in terms of X, Y and Z, the mass fraction of H, He and metals, respectively. The mass fractions $X = \rho_H/\rho = n_H m_p/\rho$, $Y = \rho_{He}/\rho = n_{He} 4m_p/\rho$ and $Z = \rho_Z/\rho = n_Z 16m_p/\rho$. Since O is the most abundant metal, we take O values for metals (this doesn't affect the results much because metals are typically a small fraction by mass). In the fully ionized state the electron number density $n_e = n_H+2n_{He}+8n_Z$ and the total number density $n = n_e+n_H+n_{He}+n_Z = 2n_H+3n_{He}+9n_Z$. The total number density is expressed as $n = \rho/(\mu m_p)$, where μ is the *mean molecular weight*. By comparing these definitions, $\mu^{-1} = 2X + 3Y/4 + 9Z/16$. One can also define $\mu_e = \rho/(n_e m_p) = 2/(1 + X)$. Recall that $X + Y + Z = 1$. Electrons contribute substantially to to pressure but not to the mass density. For a H plasma $\mu = 0.5$ and $\mu_e = 1$. For neutral H_2 molecules $\mu = 2$.

Scaling relations:

4 Nuclear energy production

Gamow tunneling:

p-p, CNO, triple-alpha reactions:

Testing stellar models with observations:

Solar neutrino problem:

5 Observations of stars

HR diagram:

Main sequence, giants and dwarfs:

Eddington limit:

6 Stellar evolution

7 Stellar winds & supernovae

8 Stellar rotation & magnetic fields

9 End states of stars

Degeneracy pressure:

White dwarfs & Chandrasekhar's limit:

Neutron stars & pulsars: