IISc, Fundamentals of Astrophysics, PH 217

Solution of Problem Set 2

September 28, 2016

Problem 1

$B_\nu = 2h\nu^3/(c^2[e^{h\nu/kT} - 1])$ denotes the specific intensity (energy emitted per unit normal area per unit time per frequency per unit solid angle) of a black body radiator. Calculate the frequency at which the emissivity peaks for a BB as a function of temperature (call it $\nu_{\rm max}$). From B_ν , derive B_λ, the specific intensity as a function of wavelength and derive the Wien's displacement law; i.e., the maximum B_λ occurs at $\lambda_{max} = (2.9 \times 10^6/T)\,{\rm nm\,K}$. Note that $\nu_{\rm max}\,\lambda_{\rm max} = c$!.

This problem can be found in any standard Statistical physics text book or the books that have discussed the radiative processes. See equations 1.56 and 1.57, and the associated text in the book - Rybicki & Lightman, 1979, Wiley-VCH.

Problem 2

Stefan-Boltzmann law says that the total radiative energy emitted by a BB per unit area per unit time is $\sigma\,T^4$ (σ is called the Stefan-Boltzmann constant). From B_ν express σ in terms of k, c and h. Energy density in BB radiation (energy per unit volume, integrated over all frequencies, solid angles) is expressed as $u = a\,T^4$, where a is known as radiation constant. Find the relation between a and σ.

Refer to equations 1.44 and 1.58, and the associated text in Rybicki & Lightman, 1979, Wiley-VCH.
Note: In estimation of total flux, should not be confused with as usual limit of θ in spherical polar coordinate. Here origin is chosen on the surface of sphere, hence the lower and upper limit of θ is 0 and $\pi/2$ (not π) respectively.

Problem 3 : A. R. Choudhuri/Astrophysics for physicists/Ch. 2

1. **(Problem 2.2) Consider a 'pinhole camera' having a small circular hole of diameter d in its front and having a 'film' at a distance L behind it. Show that the flux F_ν at the film plane is related to the intensity $I_\nu(\theta, \phi)$ in following way.**

$$F_\nu = \frac{\pi\,\cos^4\theta}{4f^2}\,I_\nu(\theta,\phi)\ ,$$

where $f = L/d$ is the 'focal ratio'.

The solid angle subtended by the area ΔA at the pin hole is

$$\Delta\Omega = \frac{\Delta A \cos\theta}{r^2} \tag{1}$$

Here $\Delta A = \pi(d/2)^2$, $\cos\theta = L/r$ and θ is the incident angle . Therefore, the flux at the film plane is

$$F_\nu = \int I_\nu \cos\theta d\Omega \simeq I_\nu \cos\theta \Delta\Omega \approx \frac{\pi \cos^4\theta}{4f^2} I_\nu(\theta,\phi) \tag{2}$$

2. **(Problem 2.5) Consider a spherical cloud of gas with radius and a constant inside temperature T far away from the observer.**
   (a) Assuming the cloud to be optically thin, find out how the brightness seen by the observer would vary as a function of distance b from the cloud center.
   (b) What is the overall effective temperature of the cloud surface?
   (c) How will the answers to (a)-(b) be modified if the cloud were optically thick.

   Refer to solved problem 1.8 in the book - Rybicki & Lightman, 1979, Wiley-VCH

3. **(Problem 2.7) Consider an atmosphere of completely ionized hydrogen having the same density as the density of Earth's atmosphere. Using the fact that a beam of light passing through this atmosphere will be attenuated due to Thomson scattering by free electrons, calculate the path length which this beam has to traverse before its intensity is reduced to half its original strength. (This problem should give you an idea of why the matter-radiation decoupling to be discussed in $ 11.7 took place after the number of free electrons was reduced due to the formation of atoms.)**

   For a strict solution of this problem one would need to treat true absorption differently from pure scattering (see $ of the book - Rybicki & Lightman). However, for most cases where the absorption coefficient (α) is lesser than the scattering coefficient (σ), the extinction coefficient is dominated by the scattering cross-section. The mean free path of a photon is thus given by $\lambda = \frac{1}{\sigma+\alpha} \simeq \frac{1}{\sigma}$. For electrons with Thomson scattering cross-section σ_T, the scattering coefficient (which is different from the scattering cross-section) is given by $\sigma = n_e\sigma_T$, where n_e is the density of electrons. The path length at which a beam of light is reduced to half its intensity is given by

$$I_{1/2} = 0.5I_0 = I_0 e^{-l_{1/2}/\lambda}$$

   The electron density is evaluated by taking the atmosphere's mass density $\rho = 1.2kg/m^3$ and assuming it is composed entirely of hydrogen atoms (of mass $\sim$ mass of a proton m_p). Putting the values of earth's particle density $n_e = \rho/m_p = 7.1 \times 10^{26} m^{-3}$ and Thomson scattering cross section $\sigma_T = 6.65 \times 10^{-29} m^2$, gives a value for the path length $l_{1/2} = 14.5m$.

   Thus in 14.5m, half of the intensity of the beam is lost and in about 100 m, more than 99% is lost. In the early universe, when most of the atoms were in an ionized state, the radiation would interact with the free electrons and would not be able to penetrate through. Once atoms started forming and the number of free electrons became lesser, the radiation was able to escape and the universe, in a sense became transparent. The cosmic microwave background radiation that we see today is composed of the escaped photons from that era and the virtual surface from which these photons emerge is called the *surface of last scattering.*

Problem 4 : Dan Maoz/Astrophysics in a Nutshell/Ch. 2

1. **(Problem 2.1)**
   a. If the sun subtends a solid angle Ω on the sky, and the flux from the Sun just above the Earth's atmosphere, integrated over all wavelengths is $f(d_\odot)$, show that the flux at the Solar photosphere is $\pi f(d_\odot)/\Omega$.
   b. The angular diameter of the Sun is 0.57 degree. Calculate the solid angle subtended by the Sun, in steradians.
   c. The Solar flux at Earth is

$$f(d_\odot) = 1.4 \times 10^6 \ \mathrm{erg\,s^{-1}\,cm^{-2}} = 1.4 \ \mathrm{kW\,m^{-2}}$$

   Use (b), and the Stefan-Boltzmann law, to derive the effective surface temperature of the Sun.
   d. Derive an expression for the surface temperature of the Sun, in terms only of its solid angle, its flux per unit wavelength $f_\lambda(\lambda_1)$ at Earth at one wavelength λ_1, and fundamental constants.

a. Assuming that the Sun emits isotropically at a luminosity $L_\odot$, the flux at a given distance R from the sun would be $f(d) = \frac{L_\odot}{4\pi d^2}$. The ratio of flux at the solar photosphere $f(R_\odot) = F_\odot$ to the flux at the Earth's atmosphere $f(d_\odot)$ would be $F_\odot/f(d_\odot) = R_\odot^2/d_\odot^2$. The solid angle subtended by the sun at Earth's surface (see Fig. 1) is given by $\Omega = \pi R_\odot^2/d_\odot^2$. This and the previous expression combine to give $F_\odot = f(d_\odot)\pi/\Omega$.

b. The radius of the sun $R_\odot$ can be expressed in terms of its angular diameter (2α) by $R_\odot \simeq \alpha d_\odot$. Combining this with the expression for Ω above, gives

$$\Omega \simeq \pi\alpha^2 = \pi(0.57/2\pi/180)^2 = 7.8 \times 10^{-5} \text{ steradians.}$$

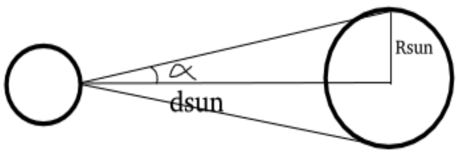

Figure 1: Solid angle subtended by sun at earth.

c. As per Stefan-Boltzmann Law, the flux of sun would be $F_\odot = \sigma T_E^4$. Combining this and the expression from (a) gives

$$T_E^4 = \frac{\pi f(d_\odot)}{\Omega\sigma} \ ; \ T_E \simeq 5800K$$

d. To relate $f_\lambda(\lambda_1)$ with T, we assume the Sun's surface to be an isotropically emitting blackbody, i.e its specific intensity is $I_\lambda = B_\lambda(T)$. Thus the flux at Sun's surface for a given wavelength would be $F_\lambda(\lambda_1) = \pi B_\lambda(T)$ (see equation 1.14 of Rybicki & Lightman for the expression for flux of an isotropically emitting body). Combining this with the expression in (a), gives

$$f_\lambda(\lambda_1) = \pi^2/\Omega B_\lambda(T)$$

Putting the expression for the Planck function $B\lambda(T)$ gives the required expression for T in terms of $f_\lambda(\lambda_1)$.

2. **(Problem 2.2)**
a. Show that, if the ratio of the blackbody fluxes from a star at two different frequencies (i.e., a color) is measured, then, in principle, the surface temperature of the star can be derived, even if the stars solid angle on the sky is unknown (e.g., if it is too distant to be spatially resolved, and its distance and surface area are both unknown).
b. Explain why it will be hard, in practice, to derive the temperature measurement if both frequencies are on the Rayleigh-Jeans side of the blackbody curve, $h\nu << kT$.
c. For the case that both measurements are on the Wien tail of the blackbody curve, $h\nu >> kT$, derive a simple, approximate, expression for the temperature as a function of the two frequencies and of the flux ratio at the two frequencies.
d. If, in addition to the flux ratio in (c), a parallax measurement and the total flux (integrated over all frequencies) at Earth are available, show that the stars radius can be derived.

a. If the two blackbody fluxes are written as $B_1(T)$ and $B_2(T)$ respectively, then their ratio $B(T)$ can be written as;

$$B(T) = \left(\frac{\nu_1}{\nu_2}\right)^2 \frac{\exp(h\nu_2/kT) - 1}{\exp(h\nu_1/kT) - 1} \tag{3}$$

Solving the above equation numerically can give in principle the surface temperature of the star.
b. In the Rayleigh-Jean's limit, $h\nu << kT$ and therefore, the exponential term can be expanded to have

$$B(T) = \left(\frac{\nu_1}{\nu_2}\right)^2 \frac{(1 + h\nu_2/kT - 1)}{(1 + h\nu_1/kT - 1)} = \frac{\nu_1}{\nu_2} \tag{4}$$

From the final equation, T is missing which confirms the fact that it will be difficult to estimate the surface temperature of the star in the Rayleigh-Jean's limit.
c.In the Wien's limit, $h\nu >> kT$ and thus, the exponential terms are much greater than 1 each and the equation can be written as;

$$B(T) = \left(\frac{\nu_1}{\nu_2}\right)^2 \frac{\exp(h\nu_2/kT)}{\exp(h\nu_1/kT)} \tag{5}$$

Proper evaluation of the above equation will give;

$$T = \frac{h}{k\, ln(B(\frac{\nu_2}{\nu_1})^2) - (\nu_2 - \nu_1)} \tag{6}$$

where B is the ratio of B_1 to B_2
d. From the parallax measurement, the distance d to the star can be determined and since the flux at Earth is known, the luminosity L can be written as

$$L = 4\pi d^2 F \tag{7}$$

On the other hand, we know that the luminosity of a spherical object radiating as a blackbody is;

$$L = 4\pi R^2 \sigma T^4 \tag{8}$$

Thus, from the two equations above, the radius R of the star can be measured.

3. (Problem 2.3) If parallax can be measured with an accuracy of 0.01 arc second, and the mean density of stars in the Solar neighbourhood is 0.1 pc^{-3}, how many stars can have their distances measured via parallax?

Distance covered in 0.01 arc second is

$$d = \frac{1\mathrm{AU}}{0.01\,\mathrm{arc\,sec}} \approx 100^3 \tag{9}$$

If the mean number density of stars is $n = 0.1\,\mathrm{pc}^{-3}$, then total number of stars is

$$N = \frac{4}{3}\pi d^3\, n = \frac{4}{3}\pi (100)^3\, 0.1 = 4.19 \times 10^5 \tag{10}$$

4. (Problem 2.4) The maximal radial velocities measured for the two components of a spectroscopic binary are 100 km s^1 and 200 km s^1, with an orbital period of 2 days. The orbits are circular.
a. Find the mass ratio of the two stars.
b. Use Kepler's Law (Eq. 2.42) to calculate the value of $M\,\sin i$ for each star, where M is the mass and i is the inclination to the observer's line of sight of the perpendicular to the orbital plane.
c. Calculate the mean expectation value of the factor $\sin^3 i$, i.e., the mean value it would have among an ensemble of binaries with random inclinations. Find the masses of the two stars, if $\sin^3 i$ has its mean value.

Applying Kepler's third law in the binary system, we have

$$P = \frac{2\,\pi}{\sqrt[2]{G(M_1 + M_2)}} a^{3/2} \quad , \tag{11}$$

where P is the orbital period of the binary system, a is the size of semi major axis, and M_1 and M_2 are the mass of the stars. If a_1 is the distance of star 1 (mass M_1) from the center of mass, then $a_1 = [M_2/(M_1 + M_2)]\,a$. Similarly for star 2, $a_2 = [M_1/(M_1 + M_2)]\,a$. Using these equation from Eq. 11 we have

$$f_1 = \frac{M_1^3}{(M_1 + M_2)^2} \sin^3 i = \frac{P}{2\pi G}(v_2 \sin i)^3 \tag{12}$$
$$f_2 = \frac{M_2^3}{(M_1 + M_2)^2} \sin^3 i = \frac{P}{2\pi G}(v_1 \sin i)^3 \tag{13}$$

Here $f_1\,, f_2$ are called the mass function of the stars 1 and 2 respectively.

a. Taking ratio f_1/f_2 we have $M_1/M_2 = v_2/v_1$. The given value of $v_{1\mathrm{obs}} = v_1\,\sin i = 100\,\mathrm{km\,s^{-1}}$ and $v_{2\mathrm{obs}} = v_2\,\sin i = 200\,\mathrm{km\,s^{-1}}$. Therefore, the mass ratio of the two stars is $M_1 : M_2 = 2 : 1$.

b. By taking $M_2 = 0.5\,M_1$, from Eq. 12, we have $M_1 \sin^3 i = 2.25\frac{P}{2\pi G}(|v_{2\mathrm{obs}}|)^3$. Now putting the values of $v_{1\mathrm{obs}}$ and $P = 2\,\mathrm{days}$, we get $M_1 \sin^3 i = 3.71\,\mathrm{M}_\odot$ and $M_2 \sin^3 i = 1.85\,\mathrm{M}_\odot$.

c. The solid angle is defined as $d\Omega = dA/r^2 = \sin i\, di\, d\phi$. Therefore, the mean expectation value of $\sin^3 i$ is

$$\left\langle \sin^3 i \right\rangle_\Omega = \frac{\int_{i=0}^{i=\pi/2}\int_{\phi=0}^{\phi=2\pi} \sin^4 i\, di\, d\phi}{\int_{i=0}^{i=\pi/2}\int_{\phi=0}^{\phi=2\pi} \sin i\, di\, d\phi} = \frac{1}{2}\beta(5/2, 1/2) = \frac{\Gamma(5/2)\,\Gamma(1/2)}{2\,\Gamma(3)} = \frac{3\pi}{16} \tag{14}$$

(Note: $i = 0$ and $i = \pi/2$ are the face-on and edge-on view of the orbit. Don't be confused with as usual limit of i (or θ) in spherical polar coordinate.)
Considering $\sin^3 i$ has its mean value, from (b) we get $M_1 = 6.29\,\mathrm{M}_\odot$ and $M_2 = 3.15\,\mathrm{M}_\odot$.

5. **(Problem 2.5) In an eclipsing spectroscopic binary, the maximal radial velocities measured for the two components are $20\ \mathrm{km\,s^{-1}}$ and $5\ \mathrm{km\,s^{-1}}$. The orbit is circular, and the orbital period is $P = 5$ yr. It takes 0.3day from the start of the eclipse to the main minimum, which then lasts 1 day.**
a. Find the mass of each star. Since the binary is of the eclipsing type, one can safely approximate $i \approx 90^o$. Check to what degree the results are affected by small deviations from this angle, to convince yourself that this is a good approximation.
b. Assume again $i = 90^o$ and find the radius of each star. Is the result still insensitive to the exact value of i.

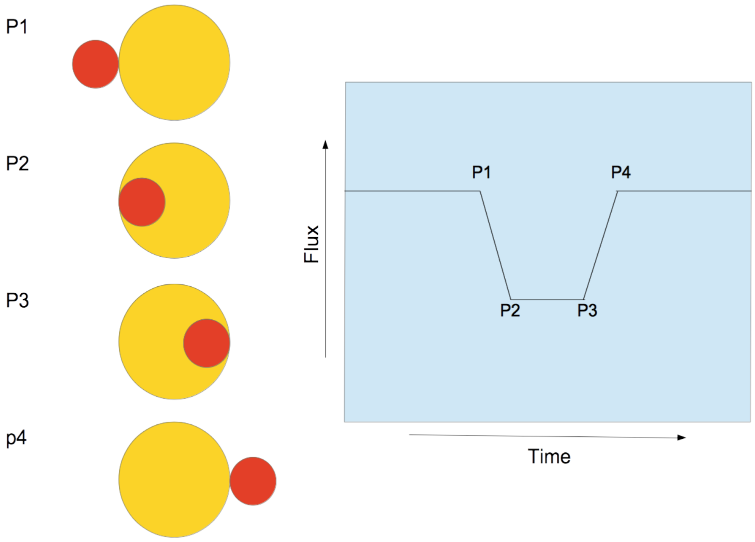

Figure 2: Schematic diagram of eclipsing binary. Position P1 represents the starting point of eclipse. The main minimum starts from P2 and it lasts upto P3.

a. The mass ratio of this binary system is $M_1/M_2 = v_2/v_1$, where $v_{1\mathrm{obs}} = v_1\ \sin i = 20\,\mathrm{km\,s^{-1}}$ and $v_{2\mathrm{obs}} = v_2\ \sin i = 5\,\mathrm{km\,s^{-1}}$. This gives $M_1 : M_2 = 1 : 4$.
Using $M_1 : M_2 = 1 : 4$ and $P = 5$ yr, from Eq. 12, we get $M_1 \sin^3 i = 25\frac{P}{2\pi G}(|v_{2\mathrm{obs}}|)^3 = 0.58\ \mathrm{M}_\odot$ and $M_2 \sin^3 i = 100\frac{P}{2\pi G}(|v_{1\mathrm{obs}}|)^3 = 2.35\ \mathrm{M}_\odot$.

b. The relative velocity of the stars is $v_{rel} = v_1 + v_2 \approx (|v_{1\mathrm{obs}}| + |v_{2\mathrm{obs}}|) \approx (20+5) = 25\ \mathrm{km\,s^{-1}}$. From Figure 2, we find that the diameter of star 1 is $d_1 = 0.3\,\mathrm{days} \times v_{rel} = 0.925\,r_\odot$, and diameter of star 2 is $d_2 = 1.3\,\mathrm{days} \times v_{rel} = 4.01\,r_\odot$. Hence the radius of the stars is $r_1 = 0.46\,r_\odot$ and $r_2 = 2\,r_\odot$ respectively.

Problem 5 :

Assuming spherical symmetry, constant opacity, and the fact that temperature increases with depth inside the sun, argue that the edge of the solar disk will be

darker compared to the center. This is the well-known limb darkening.

On the one hand, the sun is not a perfect blackbody and as a result, the temperature is not perfectly same everywhere on the surface. More importantly, the radiation we observe from the sun comes from the region with same value of optical depth (photosphere where $\tau \sim 1$).

If observer looks at the limb (or edge) of the sun rather than the center at some elevation θ as depicted in the first diagram below, the radiation is attenuated by a factor τ and the corresponding temperature within the star is given by

$$T(\tau) = T_o + T\tau\cos\theta \tag{15}$$

T_o is the brightness temperature at the surface of the sun which increases as one goes deeper into the sun. Therefore for the same value of τ (of the order 1), the brightness temperature seen by looking through the sun's photosphere from its limbs is attenuated by a factor $\cos\theta$ i.e,

$$T = T_o + T\tau\cos\theta \tag{16}$$

Whereas, if the same observer looks through the center (a distance L) of the sun rather that its limbs, he sees through to a hotter part of the sun with temperature given as

$$T = T_o + T \tag{17}$$

For this reason, the limbs or edges of the sun appear darkened. This is depicted in the figures below. Refer to the url `http://www.iucaa.in/~dipankar/ph217/contrib/limb.pdf` for more details.

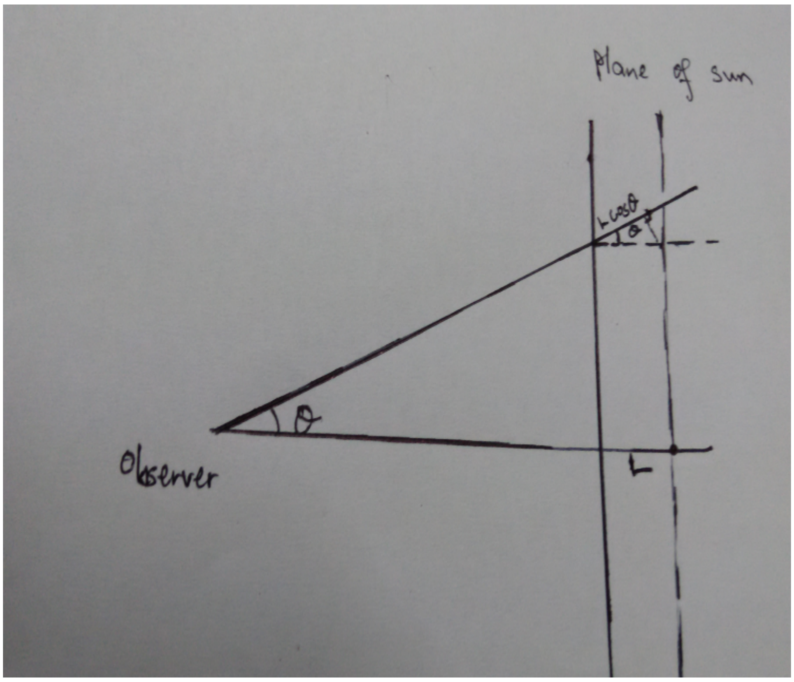

Figure 3: Schematic diagram showing limb darkening effect when looking through the edges of the sun, L here represent the region within the sun that the observer can see through (i.e, $\tau \sim 1$) which is the photosphere.

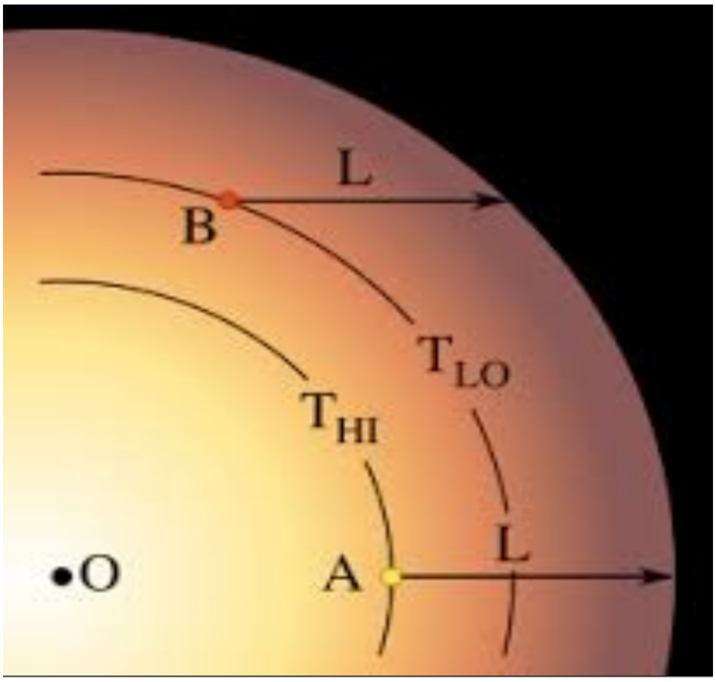

Figure 4: Limb darkening of sun. T_{LO} is a lower temperature than T_{HI}, the observer who looks through point A sees a hotter region than the observer who looks through the limbs at point B.