IISc, Fundamentals of Astrophysics, PH 217

Solution of Problem Set 3

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Problem 1

The density in the center of the sun is about 100 times that of water in normal conditions. While we cannot treat water as an ideal gas at room temperature, show by comparing the average kinetic energy of particles at the central temperature of the sun (10^7 K) that the ideal gas approximation is a good one.

Let l be the mean distance between two particles. If n is the particle number density then $l = n^{-1/3}$ i.e., $l \simeq (\rho/m_{\rm H})^{-1/3}$. For sun $\rho \approx 100$ g cm^{-3} which gives $l = 2.56 \times 10^{-9}$ cm (for water $l = 1.2 \times 10^{-8}$ cm).

If $l >> \lambda$, then each particle in the gas is distinguishable and it can be treated as ideal gas (λ is the thermal de-Broglie wave length). We know $\lambda = h/p = h/\sqrt{2\,m\,E_{\rm k}} = h/\sqrt{2\pi\,m\,k_{\rm B}T}$ (where $m \approx m_{\rm H}$). In case of sun, $\lambda = 5.5 \times 10^{-11}$ cm i.e., $\lambda << l$ (for water $\lambda = 1 \times 10^{-8}$ cm) which shows each particles are distinguishable.

Note that the particles in an ideal gas interact collisionally, and therefore we need to check an additional criterion to fulfil the ideal gas assumption. To do this, we have to prove that the average kinetic energy of the particles ($E_{\rm K}$) >> electrostatic potential ($E_{\rm P}$). For sun

$$E_{\rm K} \sim k_{\rm B}T = 1.38 \times 10^{-16} \times 10^7 = 1.38 \times 10^{-9} \text{ erg} ,$$

$$E_{\rm P} \sim \frac{e^2}{l} = \frac{\left(4.8 \times 10^{-10}\right)^2}{2.56 \times 10^{-9}} = 9 \times 10^{-11} \text{ erg}$$

This shows $E_{\rm K} >> E_{\rm P}$, and therefore, the ideal gas approximation is quite good.

Problem 2

Using order of magnitude estimates, argue that the central/virial temperature of the sun (T_c) and the effective temperature are related to $\lambda_{\rm mfp}/R$ (λ is an estimate of the photon mean free path) as $\lambda_{\rm mfp}/R \sim (T_{\rm eff}/T_c)^4$.

We know $L = 4\pi R^2\,\sigma\,T_{\rm eff}^4$, where L is the luminosity of the star. The mean free path $\lambda_{\rm mfp} \approx 1/(\sigma_{\rm T} n)$. Therefore

$$\lambda_{\rm mfp} = \frac{4\pi\mu m_{\rm H}}{L}\left(\frac{R^2}{\rho}\right)\,T_{\rm eff}^4 \qquad (1)$$

From Virial theorem we have $T_c = \frac{\mu m_H}{k_B} \frac{GM}{R}$. Rearranging we get

$$\lambda_{\rm mfp}/R \propto (T_{\rm eff}/T_c)^4$$

Problem 3

Assuming that nuclei A and B have a Maxwellian distribution of velocities in equilibrium at temperature T , show that the 1-D velocity distribution of relative velocities ($v = |v_A - v_B|$) is also a Maxwellian given by

$$f(v) = \left(\frac{\mu}{2\,\pi\,k_B\,T}\right)^{3/2}\ 4\pi v^2\ \exp\left(-\frac{\mu\,v^2}{2\,k_B\,T}\right) \tag{2}$$

and $\mu = m_A m_B/(m_A + m_B)$ is the reduced mass. Express this velocity distribution as an energy ($E = \mu\,v^2/2$) distribution. Show that the typical thermal velocity of massive particles is smaller compared to lighter ones.

This problem is quite straight forward and it can be found in any standard Statistical physics text book.

Problem 4

For a non-relativistic degenerate electron gas for which $p \propto \rho^{5/3}$ (p is pressure and ρ is mass density). Using scaling arguments, show that the radius of a non-relativistic white dwarf which is supported by electron degeneracy scales with its mass as $R \propto M^{-1/3}$ and $\rho \propto M^{-1/2}$. This is different from normal stars, flow which we argued (again using scaling arguments) that $R \propto M$. Degeneracy pressure starts to become important when de Broglie wavelength becomes of order the mean separation between electrons. Calculate the order of magnitude radius of a white dwarf for which de Broglie wavelength (assuming viral temperature) becomes equal to the mean distance between electrons. What happens to the above estimates if electrons become relativistic?

The hydrostatic equilibrium equation leads to

$$\frac{1}{\rho}\frac{dP}{dr} = -\frac{G\,M(r)}{r^2} \tag{3}$$

$$\frac{d}{dr}\left(\frac{r^2}{\rho}\frac{dP}{dr}\right) = -G\frac{dM}{dr}$$

Now using $dM/dr = 4\pi r^2 \rho$, $P = K\rho^{(1+1/n)}$ and $\rho = \rho_c\,\theta^n$ we get

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{d\theta}{dr}\right) = -\left[\frac{4\pi G\rho_c^{1-1/n}}{(n+1)K}\right]\theta^n$$

The above equation is well known Lane-Emden equation. The mass of the white dwarf can be estimated as follows :

$$M = \int_0^R 4\pi r^2 \rho dr = \int_0^R 4\pi r^2 \rho_c \theta^n dr = -\left[\frac{(n+1)K\rho_c^{1/n}}{G}\right]\left(r^2\frac{d\theta}{dr}\right)_0^R$$

At the center ($r = 0$) of a star $\theta = 1$ and $d\theta/dr = 0$. On the surface ($r = R$) $\theta = 0$. Therefore we get

$$M = \left[\frac{(n+1)K\rho_c^{1/n}}{G}\right]\left[r^2\frac{d\theta}{dr}\right]_{r=R} \propto \rho_c^{1/n} R^2 \frac{1}{R} \tag{4}$$

where we can use $\rho_c \sim M/R^3$. Therefore we get

$$R = \left[\frac{G}{(n+1)K}\right]^{1/(n-3)} M^{(n-1)/(n-3)} \tag{5}$$

For non-relativistic star, $P = K\rho^{5/3}$ i.e., $n = 3/2$ which gives $R \propto M^{-1/3}$.

The de-Broglie wave length $\lambda = h/p$. For relativistic electron $p \simeq m_e c$ which yields $\lambda = h/(m_e\,c)$. The mean separation between two particles $l \simeq n^{-1/3}$. Considering at $\lambda = l$ the density $\rho = \rho_{\rm deg}$ which gives

$$\rho_{\rm deg} \approx m_{\rm H} \left(\frac{m_e\,c}{h}\right)^3 = 1.12 \times 10^5 \ {\rm g\,cm^{-3}}$$

For relativistic electrons $P = K\rho^{4/3}$ which gives $R \propto M^{2/0}$, i.e. Mass and radius do not depend on the central density!

Problem 5

Using Paulis exclusion principle, argue that the number protons and neutrons in a nucleus is roughly equal and that neutrons in a nucleus simply cannot decay via beta decay. Also, why do massive nuclei have over abundance of neutrons over protons (i.e., $A > 2Z$)? Hint: unlike neutrons, protons are charged.

For most of the low mass stable nuclei, the proton number (Z) is equal to the neutron number (N). This is because, neutrons are charge neutral and they interact only via nuclear force which is attractive in nature. Whereas, protons interact via both nuclear force and coulomb force. For small number of protons, the equal number of neutrons are enough to hold the protons together. Whereas for heavy nucleus, to overcome the proton-proton repulsion, excess number of neutrons are required.

According to Paulis exclusion principle, each nuclear energy level contains two nucleons of opposite spin. When both Z and N are even (e-e nucleus; i.e., when the energy levels are filled), the nucleus doesn't want to gain or lose nucleons by participating in nuclear reactions. The e-e nucleus are quite stable, even when they are heavy. When Z and N both are odd, the nuclear force between N and Z keeps them stable. The most unstable case is when either Z or N are odd (e-o of o-e nucleus) and they decay via beta decay.

Problem 6 : A. R. Choudhuri/Astrophysics for physicists/Ch. 3

1. **(Problem 3.2) If the Sun was producing its energy by slow contraction as suggested by Helmholtz and Kelvin, estimate the amount by which the radius of the Sun has to decrease every year to produce the observed luminosity.**

   Helmholtz and Kelvin proposed that the main source of power of a star is the gravitational energy. The gravitational energy of a star can be written as $E = a\,(GM^2/R)$, where a depends on the mass distribution. According to Kelvin and Helmholtz proposal, $|dE/dt| = L$, where L is the observed luminosity of the star. Assuming $a \sim 1$, we obtain $dE/dt = -(GM^2/R^2)dR/dt$. Therefore we get

   $$\frac{dR}{dt} = -\frac{L\,R^2}{GM^2}$$

   Using $R = 6.96 \times 10^{10}$ cm, $L = 3.846 \times 10^{33}\,{\rm erg\,s^{-1}}$ we get $\frac{dR}{dt} = 3.17 \times 10^{-8}\,{\rm R_\odot\,yr^{-1}}$.

2. **(Problem 3.3) Show that the radiation pressure at the centre of the Sun is negligible compared to the gas pressure, by estimating the ratio of the radiation pressure to the gas pressure.**

The radiation pressure of a blackbody radiation is given by $P_{\rm rad} = (1/3)\, a T^4$, where $a = 7.6 \times 10^{-15}\,{\rm erg\,cm^{-3}\,K^{-4}}$. The gas pressure $P_{\rm gas} \approx (\rho/m_{\rm H})\, k_{\rm B}\, T$. This gives

$$\frac{P_{\rm rad}}{P_{\rm gas}} = \frac{(1/3)\, a\, T^4}{(\rho/m_{\rm H})\, k_{\rm B}\, T} \approx 3 \times 10^{-4}$$

This shows $P_{\rm rad} << P_{\rm gas}$.

3. **(Problem 3.7) Make a very rough estimate of the wavelengths at which a star of mass 9 $M_\odot$ and a star of mass 0.25 $M_\odot$ will give out maximum radiation.**

From the definition of effective temperature we have $T_{\rm eff} \propto (L/R^2)^{1/4}$. For main sequence star, $L \propto M^4$ and $R \propto M$ which yield $T_{\rm eff} \propto M^{1/2}$ i.e., $T_{\rm eff} = \alpha M^{1/2}$ (α is proportional constant). We can use the standard parameters of the sun to estimate the value of α which gives

$$T_{\rm eff} = T_\odot \left(\frac{M}{M_\odot}\right)^{1/2}$$

Using Wien's displacement law $\lambda_{\rm max} \simeq (0.2898/T_{\rm eff})\,{\rm cm\,K}$, we get

$$\lambda_{\rm max} = \left(\frac{0.2898}{T_\odot}\right)\left(\frac{M}{M_\odot}\right)^{-1/2} {\rm cm} = 501.6 \left(\frac{M}{M_\odot}\right)^{-1/2} {\rm nm}$$

For a star of 9 $M_\odot$, $\lambda_{\rm max} = 167.2\,{\rm nm}$. Similarly for $M = 0.25\ M_\odot$, $\lambda_{\rm max} = 1003.2\,{\rm nm}$.

4. **(Problem 4.3) According to current solar models, the centre of the Sun has a temperature of about 1.56×10^7 K, a density of about $1.48 \times 10^5\ {\rm kg\,m^{-3}}$ and a chemical composition given by $X_{\rm H} = 0.64$, $X_{\rm He} = 0.34$, $X_{\rm CNO} = 0.015$. Estimate the amount of energy that is generated per unit volume at the centre of the Sun due to the pp chain and the CNO cycle.**

Note : Please see Equation 4.25 and 4.27 in Chapter 4 of this book (Astrophysics for physicists by A. R. Choudhuri).

5. **(Problem 4.4) Make a very rough estimate of the time that an acoustic wave propagating radially inward in the Sun would take to go from one end of the Sun to the other end.**

The speed of sound near the center of the sun $c_{\rm s} = \sqrt{k_{\rm B}T/(\mu\, m_{\rm H})} \approx 371\,{\rm km\,s^{-1}}$ and near the surface $c_{\rm s} \approx 9\,{\rm km\,s^{-1}}$. For a rough estimation, let us assume $c_{\rm s} \approx 100\,{\rm km\,s^{-1}}$ and it is independent of radius. Therefore, the acoustic wave passing time scale is $t_\odot = R_\odot/c_{\rm s} \sim 0.08$ day.

Note : Acoustic wave propagation in the Sun is an interesting field and the branch is known as Helioseismology (for more details see http://surya.as.utexas.edu/helio.html).

6. **(Problem 4.6) Neutrinos from Supernova 1987A which reached the Earth travelling a distance of 55 kpc were found to have energies in the range $6-39$ MeV. If the spread of 12 s in arrival times was caused by neutrinos of different energies travelling at different speeds, show that the neutrino mass cannot be much more than about 20 eV**

Let us assume all neutrinos were produced in a single event (eg., Supernova) and due to the difference in their initial energies, they have arrived at different times. Consider

$E_{\rm H} = \gamma_{\rm H} m\, c^2$ and $E_{\rm L} = \gamma_{\rm L} m\, c^2$ denote the energy of the earliest and latest neutrinos respectively i.e., $E_{\rm H} = 39$ MeV and $E_{\rm L} = 9$ MeV.

If each neutrinos travel a distance D, then the time taken by the high energy neutrino to reach the earth is $t_{\rm H} = D/v_{\rm H}$, where $v_{\rm H} = c\sqrt{1 - 1/\gamma_{\rm H}^2}$ is the speed of the high energy neutrino. This gives $t_{\rm H} = t_0/\sqrt{1 - 1/\gamma_{\rm H}^2}$. Again assuming $\gamma_{\rm H} >> 1$, we obtain $t_{\rm H} \simeq t_0 \left[1 + 1/(2\,\gamma_{\rm H}^2)\right]$. Similarly for the low energy neutrino, $t_{\rm L} \simeq t_0 \left[1 + 1/(2\,\gamma_{\rm L}^2)\right]$. This gives

$$t_{\rm L} - t_{\rm H} = \frac{t_0}{2}\left[\frac{1}{\gamma_{\rm L}^2} - \frac{1}{\gamma_{\rm H}^2}\right]$$

$$\gamma_{\rm L} = \left[\frac{t_0\left\{1 - (E_{\rm L}/E_{\rm H})^2\right\}}{2\,(t_{\rm L} - t_{\rm H})}\right]^{1/2} \tag{6}$$

Using all given values we get $\gamma_{\rm L} = 4.5 \times 10^5$. Therefore, $m_{\rm max}\, c^2 = E_{\rm L}/\gamma_{\rm L} = 20$ eV.

Note : See the discovery paper :
1. http://journals.aps.org/prl/cited-by/10.1103/PhysRevLett.58.1490
2. http://journals.aps.org/prl/abstract/10.1103/PhysRevLett.58.2722

Problem 7 : Dan Maoz/Astrophysics in a Nutshell/Ch. 3

1. **(Problem 3.4) Consider a hypothetical star of radius R, with density that is constant, i.e., independent of radius. The star is composed of a classical, no-relativistic, ideal gas of fully ionized hydrogen.**
   a. Solve the equations of stellar structure for the pressure profile, $P(r)$, with the boundary condition $P(R) = 0$.
   b. Find the temperature profile, $T(r)$.
   c. Assume that the nuclear energy production rate depends on temperature as $\epsilon \propto T^4$. (This is the approximate dependence of the rate for the pp chain at the temperature in the core of the Sun.) At what radius does ϵ decrease to 0.1 of its central value, and what fraction of the stars volume is included within this radius?

a. From hydrostatic equilibrium we have

$$-\frac{1}{\rho}\frac{dP}{dr}\,\hat{r} + \vec{f}_G = 0\ , \tag{7}$$

where $\vec{f}_G = -\hat{r} G M(r)/r^2$ and $M(r) = (4/3)\pi r^3\,\rho$. Now integrating above equation we get

$$\int_{r=R}^{r} dP = -\frac{4\,\pi}{3}\,G\rho^2 \int_{r=R}^{r} dr\ r$$

$$P(r) = \frac{2\,\pi}{3}\,G\rho^2\ (R^2 - r^2) \tag{8}$$

b. From ideal gas equation $P = \rho/(\mu m_{\rm H})\,k_B\,T$. This gives

$$T(r) = \frac{\mu\, m_{\rm H}}{k_B}\,\frac{P}{\rho} = \frac{\mu\, m_{\rm H}}{k_B}\,\frac{2\,\pi}{3}\,G\rho\ (R^2 - r^2) \tag{9}$$

c. The dependence of the nuclear energy production on temperature is given as $\epsilon \propto T^4$. Using problem a and b we get

$$\epsilon \propto (R^2 - r^2)^4 = \epsilon_0 \left[1 - (r/R)^2\right]^4 \ ,$$

Let at $r = r_{\rm f}$, $\epsilon = f\,\epsilon_0$, where ϵ_0 is the central value ($r = 0$) and $f \leq 1$. Therefore,

$$r_f = R\left[1 - f^{1/4}\right]^{1/2} \tag{10}$$

Putting $f = 0.1$ we get $r = 0.66\,R$.

2. **(Problem 3.5)Suppose a star of total mass M and radius R has a density profile $\rho(r) = \rho_{\rm c}(1 - r/R)$, where is the central density.**
   a. Find $M(r)$.
   b. Express the total mass M in terms of R and $\rho_{\rm c}$.
   c. Solve for the pressure profile, $P(r)$, with the boundary condition $P(R) = 0$.

a.

$$M(r) = \int_{r=0}^{r} 4\pi\, r^2\, dr\ \rho_c(1 - r/R) = \frac{4\pi}{3}\,\rho_c\left(r^3 - \frac{3}{4}\frac{r^4}{R}\right)$$

b. Total mass $M_{\rm tot} = \frac{\pi}{3}\,\rho_c\,R^3$.
c. Following the same method done in problem 3.4.a, we get

$$\int_{r=R}^{r} dP = -4\,\pi\,G\,\rho_c^2 \int_{r=R}^{r} dr\,(1 - r/R)\left(r^3 - \frac{3}{4}\frac{r^4}{R}\right)$$

$$P(r) = \pi G\rho_c^2\,R^2\left[\frac{5}{36} - \frac{2}{3}\left(\frac{r}{R}\right)^2 + \frac{7}{9}\left(\frac{r}{R}\right)^4 - \frac{1}{4}\left(\frac{r}{R}\right)^2\right] \tag{11}$$

3. **(Problem 3.6) Consider a star of mass $M = 10\,M_\odot$, composed entirely of fully ionized ^{12}C. Its core temperature is $T_c = 6 \times 10^8$ K (compared to $T_c = 1.5 \times 10^7$ K for the Sun).**
   a. What is the mean particle mass m, in units of $m_{\rm H}$?
   b. Use the classical ideal gas law, the dimensional relation between mass, density, and radius, and the virial theorem, to find the scaling of the stellar radius $r$with total mass M, mean particle mass m, and core temperature T_c. Using the values of these parameters for the Sun, derive the radius of the star.
   c. If the luminosity of the star is $L = 10^7\,L_\odot$, what is the effective surface temperature?
   d. Suppose the star produces energy via the reaction

$$^{12}C +^{12} C \rightarrow^{24} Mg$$

The atomic weight of ^{12}C is 12, and that of 24 Mg is 23.985. (The atomic weight of a nucleus is defined as the ratio of its mass to 1/12 the mass of a ^{12}C nucleus). What fraction of the stars mass can be converted into thermal energy?
e. How much time does it take for the star to use up 10% of its carbon?

a. When ^{12}C is fully ionized, it contributes 1+6 particles (one nucleus and six electrons). Therefore, the mean mass per particle is

$$m_{\rm av} = \frac{m_{\rm nucleus} + n_e\, m_{\rm e}}{n_{\rm nucleus} + n_e} \approx \frac{A\, m_{\rm H}}{n_p + n_e} = \frac{12}{7} m_{\rm H}$$

b. For ideal gas $T = (\mu\, m_{\rm H}/k_B)P/\rho$. From hydrostatic equilibrium we have

$$\frac{1}{\rho}\frac{dP}{dr} = -\frac{G\,M}{r^2}\ , \tag{12}$$

Using scaling relation of mass $\rho \sim M/R^3$, from above equation we have $P \sim G\,M^2/R^4$. Substituting this we get

$$T \sim \frac{\mu\, m_{\rm H}}{k_{\rm B}}\, G\, \frac{M}{R} \tag{13}$$

The central temperature of sun is $T \simeq 1.65 \times 10^7$ K, $M = 2 \times 10^{33}$ g which gives $R \approx (\mu\, m_{\rm H}/k_{\rm B})\, GM/T = 5.98 \times 10^{10}$ cm= $.86\, r_\odot$.

c.

$$T_{\rm eff} = \left(\frac{L}{4\pi R^2 \sigma}\right)^{1/4} = 350295 \text{ K}$$

($\sigma = 5.67 \times 10^{-5}$ erg cm^{-3} K^{-4}, $L_\odot = 3.839 \times 10^{33}$ erg s^{-1}).

d. The mass of the star is $M = 10 M_\odot$ and it is completely made of Carbon. The total number of carbon atom is $n(^{12}C) = M/(\text{the mass of } ^{12}\text{C}) = 10\text{M}_\odot/(12 \times 1.66 \times 10^{-24}\text{ g}) \simeq 10^{57}$ and the total number of reactions is $n(^{12}C)/2 = 0.5 \times 10^{57}$.

The mass difference between final and initial elements in each fusion is $\Delta M = 2 \times M(^{12}C) - M(^{24}Mg) = (24 - 23.985)\,\text{amu} \simeq 0.015\,\text{amu} = 2.49 \times 10^{-26}$ g which is converted into thermal energy. The total amount of mass converted into thermal energy is $\Delta M_{\rm Total} = \Delta M \times \text{total number of reaction} = (2.49 \times 10^{-26}) \times (0.5 \times 10^{57}) = 1.245 \times 10^{31}$ g $= 6.225 \times 10^{-3}\ M_\odot = 6.225 \times 10^{-4} M$.

e. The energy difference between final and initial elements is $\Delta E = 2 \times M(^{12}C) - M(^{24}Mg) = (24 - 23.985)\,\text{amu} \simeq 0.015 \times 931\,\text{MeV} = 2.2372 \times 10^{-5}\,\text{erg}$. The amount of energy production in burning of 10% carbon is $E_{10\%} = (0.1 n(^{12}C)/2) \times (2.2372 \times 10^{-5})\,\text{erg} = 1.12 \times 10^{51}\,\text{erg}$. Let $t_{10\%}$ be the time in which 10% carbon converted into thermal energy and assume that the luminosity L $(= 10^7\, L_\odot)$ is independent of time. This gives

$$t_{10\%} = \frac{E_{10\%}}{L} \simeq 923 \text{ yr}$$

4. **(Problem 3.9) We saw (Eq. 3.141) that, on Earth, the number flux of Solar neutrinos from the $p - p$ chain is**

$$f_{\rm neutrino} = \frac{2\, f_\odot}{26.2\,\text{MeV}} = \frac{2 \times 1.4 \times 10^6\,\text{erg}\,\text{s}^{-1}\,\text{cm}^{-2}}{26.2 \times 1.6 \times 10^{-6}\,\text{erg}} = 6.7 \times 10^{10}\,\text{s}^{-1}\,\text{cm}^{-2}.$$

Other nuclear reactions in the Sun supplement this neutrino flux with a small additional flux of higher-energy neutrinos. A neutrino detector in Japan, named SuperKamiokande, consists of a tank of 50 kton of water, surrounded by photomultiplier tubes. The tubes detect the flash of Cerenkov radiation emitted by a recoiling electron when a high-energy neutrino scatters on it.
a. How many electrons are there in the water of the detector?
b. Calculate the detection rate for neutrino scattering, in events per day, if

10^{-6} **of the Solar neutrinos have a high-enough energy to be detected by this experiment, and each electron poses a scattering cross section** $\sigma = 10^{-43}\,\mathrm{cm}^2$

a. The effective number of electrons in the detector is $n_{\mathrm{effective}} = M/(2m_{\mathrm{H}}) = 50 \times 10^9\,\mathrm{g}/(2 \times 1.6733 \times 10^{-24}) = 1.5 \times 10^{34}$.
b. Detection rate of high energy neutrino is $10^{-6} f_{\mathrm{neutrino}} \times (\sigma\, n_{\mathrm{effective}}) = 1.01 \times 10^{-4}$ per sec. Therefore, the possible number of detections in a day $1.01 \times 10^{-4}\, 24\,\mathrm{hr} \simeq 9$.