IISc, Fundamentals of Astrophysics, PH 217

Solution of Problem Set 4

December 1, 2016

Problem 1

The energy lost due to a pulsar (rotating neutron star) can be approximated by the magnetic dipole radiation formula. A sinusoidally varying magnetic dipole radiates total energy per unit time equal to $\dot{E} = B_*^2 R_*^6 \Omega_*^4 sin^2\alpha/(6c^3)$, in terms of the polar magnetic field just outside the star (B_*), stellar radius (R_*) and angle between the rotational and magnetic dipole axes (α). Assume that pulsars are rotationally powered, and that B_* and R_* are unchanged during the evolution, derive the relation between pulsar period (P), period-derivative ($\dot{P}$) and magnetic field strengths. Estimate the dipole magnetic field strength for Crab pulsar which has $\dot{E} \sim 5 \times 10^{38}$ erg s^{-1} (measured from the observations of Crab nebula powered by the pulsar) and radius of 3 km, a rotation period of 33 ms and sin $\alpha \sim 1$. What is the pulsar lifetime $P/2\dot{P}$ for Crab (you can also calculate it exactly from the $P - \dot{P} - B$ relation) ?

Ans. The radiated energy $\dot{E}$ comes at the expense of the rotational power of the neutron star. Thus the rotational energy $E = 1/2 I\Omega^2$ must reduce with time and the period $P = 2\pi/\Omega$ must increase with time. This can be estimated as

$$\dot{E} = \frac{d}{dt}\left(\frac{1}{2}I\Omega^2\right) = \frac{d}{dt}\left(2\pi^2\frac{I}{P^2}\right)$$

Since radius and mass don't change, I does not change with time. Thus

$$\dot{E} = -\frac{2\pi^2 I\dot{P}}{P^3}$$

The negative sign denotes that energy is lost from the system. Equating this with the expression for radiation from a sinusoidally varying magnetic dipole gives

$$B_*^2\Omega^4 \propto \dot{P}/P^3$$

Upon substituting value of Ω, the desired expression is obtained as

$$B_*^2 \propto P\dot{P}$$

For Crab nebula, the magnetic field can be estimated by the expression for the radiation of a magnetic dipole. Substituting the values, gives

$$B = \sqrt{\left(\frac{\dot{E}6c^3}{R_*^6\Omega^4}\right)} \simeq 8 \times 10^{12} G$$

The pulsar lifetime can be estimated by computing $\dot{P}$ from the energy lost due to rotation and assuming the moment of inertia of Crab is $2/5M_* R_*^2$, with M_* for Crab taken as $1.4M_\odot$. Thus

$$\tau = P/2\dot{P} = \pi^2 I/(P^2\dot{E}) = \frac{2M_* R_*^2 \pi^2}{P^2 \dot{E}} \sim 2 \times 10^{10}\, s$$

Problem 2

Assuming that a temperature of the atmosphere T = 10^6 K, calculate the scale height of the isothermal sphere on the surface of a neutron star. Rotation and magnetic fields are strong enough that electrostatic force close to the surface of a neutron star is much larger than the gravitational force, and a large, dense magnetosphere is formed instead of a very thin atmosphere.

Ans. The scale height of an isothermal atmosphere can be found by using the pressure balance equation for the atmosphere, which is

$$\frac{dP}{P} = -g\rho$$

On using the ideal gas law, this can be solved

$$P = \frac{\rho}{m_p} k_B T$$

where it is assumed that the atmosphere is made of hydrogen atoms. Thus

$$\frac{dP}{dr} = -g m_p P/(k_B T)$$

On solving this differential equation, we get

$$P = P_0 e^{-r/r_0}$$

where the scale height is

$$r_0 = \frac{k_B T}{m_p g}$$

For a neutron star, the gravitational field $g \sim GM_*/R_*^2$ (neglecting GR effects). Thus

$$r_0 = \frac{k_B T R_*^2}{m_p G M_*} = 6.2\text{mm}$$

Thus the scale height (of the order of a few mm) is very small for a thermally supported atmosphere. As mentioned, the neutron star atmosphere (composed mainly of plasma) is held against gravity because of the rotating strong magnetic field, which leads to the formation of a large dense magnetosphere.

Problem 3

Associating the positional uncertainty of virtual particles with Schwarzschild radius of a BH, show that a temperature of BH photosphere can be estimated as $k_B T = hc^3/(16\pi^2 G M_{BH})$. This temperature of the BH photosphere due to quantum jittering of photons (and virtual pairs) around BHs. Assuming BB radiation (and Stefan's law), calculate the luminosity of the BH of mass M. Calculate the mass of the BH which can be evaporated within Hubble time (14Gyr, i.e, total energy radiated within Hubble time equals $M_{BH}c^2$). We do not expect Hawking radiation to be important for BHs less massive than this.

As per the uncertainty principle for particles

$$\Delta p \Delta x \geq \frac{\hbar}{2}$$

where Δp is the uncertainty in momentum of the virtual particles jittering about the BH. This uncertainty / spread of momentum can be related to the thermal energy of these virtual particles by noting that for massless particles

$$k_B T = \Delta E = c \Delta p$$

The uncertainty in position is of the order of the circumference of the BH Schwarzschild sphere giving

$$\Delta x \simeq 2\pi r_s = 2\pi \frac{2GM_{BH}}{c^2}$$

Substituting these values in the uncertainty relation and putting $\hbar = h/(2\pi)$ gives

$$k_B T = \frac{hc^3}{16\pi^2 GM_{BH}}$$

For a blackbody at temperature T, the luminosity of radiation as per Stefan's law is

$$L_{BH} = \sigma T^4 A^2$$

where $A = 4\pi r_s^2$ is the emitting surface area of the BH Schwarzschild sphere. Thus

$$L_{BH} = \frac{\sigma h^4 c^8}{4096\pi^7 (GM_{BH})^2 k_B^4}$$

The energy for this radiation is derived from the black hole mass $E = M_{BH}c^2$, thus giving

$$-\frac{d}{dt}E = -c^2 \frac{dM_{BH}}{dt} = \frac{\sigma h^4 c^8}{4096\pi^7 (GM_{BH})^2 k_B^4}$$

This equation gives the rate of mass evaporation of the black hole by the differential equation

$$M_{BH}^2 dM_{BH} = -K dt$$

where K is a constant, whose value in SI units is

$$K = \frac{\sigma h^4 c^6}{4096 G^2 \pi^7 k_B^4} = 3.96 \times 10^{15}$$

Integrating the differential equation for $t = 0$, $M_{BH} = M_0$ (current) to $t = t_{ev}$, $M_{BH} = 0$ (fully evaporated) gives the time scale t_{ev} for a black hole of mass M_0 to be completely evaporated. This comes to

$$t_{ev} = \frac{M_0^3}{3K}$$

Putting t_{ev} equal to the Hubble time (14 Gyr or 4.41×10^{15} s) gives

$$M_0 = 1.7 \times 10^{11} \text{kg}$$

which is much less than the mass of the earth. Thus black holes with masses larger than this value are expected to be still existing in the universe.

Problem 4

Sketch the effective equipotential surfaces of a test particle in the potential of two

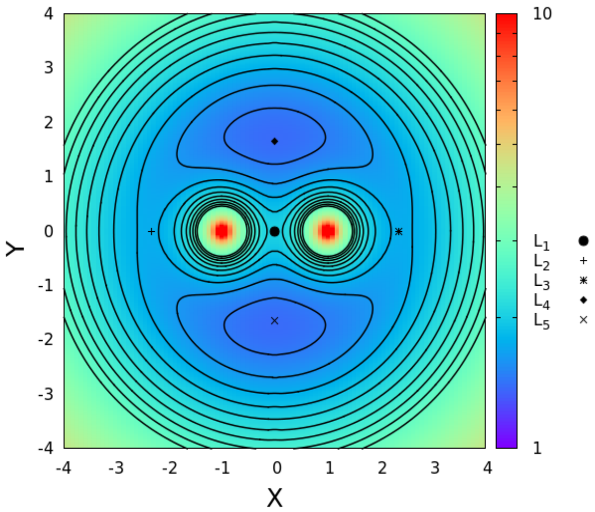

Figure 1: $\beta = 1$

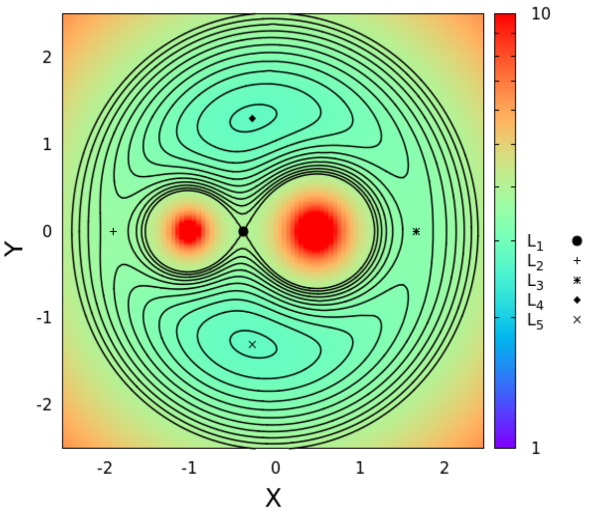

Figure 2: $\beta = 2$

rotating bodies of mass M_1 and M_2 (you can choose $M_1 = M_2$ for the plot). Mark the different equilibrium points (a computer generated contour plot is even better). How that the center of mass location is closer to the more massive body but the inner Lagrange point (L1) is closer to the less massive star. Reference: Chapter 4 in Accretion Power in Astrophysics by Frank King and Raine.

Consider a binary system where two starts of masses M_1 and M_2 are orbiting around their center of mass with an angular frequency $\Omega = \sqrt{G(M_1 + M_2)/a^3}$. Here a is the separation between them. Let p be a test object, and the distance from the respective stars is $\vec{r}_1$ and $\vec{r}_2$. The total gravitational potential of this system is

$$\Phi = -\frac{GM_1}{|\vec{r}_1|} - \frac{GM_2}{|\vec{r}_2|} - \frac{1}{2}\Omega^2 r^2 \quad ,$$

where $\vec{r}$ is the position vector of the test object. Let $\vec{r_1}$ makes an angle θ with the line joining the stars 1 and 2, and origin has been chosen at the location of the center of mass. Therefore, the above equation can be written as

$$\Phi(r,\theta) = -\frac{GM_1}{\sqrt{r^2 + a_1^2 + 2\cos\theta\, r a_1}} - \frac{GM_2}{\sqrt{r^2 + a_2^2 - 2\cos\theta\, r a_2}} - \frac{1}{2}\Omega^2 r^2 \tag{1}$$

Here $a_{1,2}$ are the distance of the stars from the origin i.e., for our case, $a_1 = aM_2/(M_1 + M_2) = a_2(M_2/M_1)$. Dividing 2 by GM_1/a_1 we obtain

$$\frac{\Phi(r,\theta)}{GM_1/a_1} = -\frac{1}{\sqrt{(r/a_1)^2 + 1 + 2\cos\theta\,(r/a_1)}} - \frac{\beta}{\sqrt{(r/a_1)^2 + (a_2/a_1)^2 - 2\cos\theta\,(r/a_1)\,(a_2/a_1)}} - \frac{\beta^3}{2(1+\beta)^2} r^2 \quad , \tag{2}$$

where $\beta = M_2/M_1 = a_1/a_2$. Figure shows the equipotential contours for $\beta = 1.0$ and $\beta = 2.0$.

The first Lagrangian point L_1 is defined as the point where gravitational force balanced by the centrifugal force. For $\beta = 1$, L_1 coincides with the center of mass (COM). For $\beta > 1$, the COM is near the heavy mass (because $a_1 = a\, M_2/(M_1 + M_2)$) but L_1 is close to the lighter mass. This is because, the gravitational force $\propto 1/r^2$, it increases when r is close to the object.

Problem 5

At high enough matter density protons+electrons, which are less massive than the mass of neutrons, can combine and form neutrons. This process is called neutron drip. We can derive the density at which neutron drip happens by equating the electron Fermi energy (this is the energy that the highest energy electrons will occupy) to the difference in the neutron and electron+proton rest mass energy. Retain the relativistic formula for Fermi energy. Convert the Fermi energy to density to calculate the neutron drip density. Above this density neutron fraction is becomes larger. See section 5.4 of Arnab's book.

Answer is given in the mentioned reference.

Problem 6

Salpeter time: Assuming that accretion on to a BH takes place at the Eddington limit. Calculate the mass of a BH that starts with a seed mass M(0) as a function of time. The growth timescale is known as Saltpeter time. What seed mass is needed at $t = 0$ to grow 10^9 $M_\odot$ BHs by red shift of 7 when the age of universe is 0.77 Gyr.

The Accretion disk of a black hole emits very energetic radiation. If the radiation power becomes greater than the accretion power then the radiation pressure stops the accretion.

Consider, at time t, the accretion rate is dM/dt. When the accreted matters fall into the black hole, their gravitational energy increases (i.e., the total energy becomes more negative). To fulfil the total energy conservation, the rest amount of energy is converted into the radiation (in addition, thermal energy, magnetic energy etc). Therefore, the luminosity of the emitted radiation (L_{ACC}) can be written as

$$L_{\mathrm{ACC}} = \frac{GM_{\mathrm{B}}(t)}{r^2}\frac{dM}{dt} \equiv \epsilon \dot{M}_{\mathrm{B}}(t)c^2\,, \tag{3}$$

where a factor ϵ is introduced to consider the uncertainties in the gas accretion (e.g., radiation pressure, ambient density profile, jet etc.). Therefore, equation 3 represents the effective rate of change of the black hole mass.

After the accretion, some fraction of the black hole mass converted into the radiation energy which effectively reduces the rest mass of the black hole. Consider the luminosity of the emitted radiation is same as the Eddington luminosity (i.e., $f_{\mathrm{ED}} \to 1$) i.e., $L_{\mathrm{ACC}} = L_{\mathrm{E}}$, where $L_{\mathrm{E}} = 4\pi G M_{\mathrm{B}} m_{\mathrm{H}} c/\sigma_{\mathrm{T}} = 1.27 \times 10^{38} (M_{\mathrm{B}}/M_\odot)\,\mathrm{erg\,s^{-1}}$ and $M_{\mathrm{B}} = (1-\epsilon)M_{\mathrm{B}}(t)$ is the mass left inside the black after losing its energy. This yields

$$\frac{dM_{\mathrm{B}}}{dt} = \left(\frac{1-\epsilon}{\epsilon}\right)\frac{1.27\times 10^{38}}{c^2}\frac{M_{\mathrm{B}}(t)}{M_\odot} \tag{4}$$

Substituting $M_{\mathrm{B}}/M_\odot$ by m_{B}, we get

$$\frac{dm_{\mathrm{B}}}{dt} = \left(\frac{1-\epsilon}{\epsilon}\right)\frac{1.27\times 10^{38}}{c^2\, M_\odot} m_{\mathrm{B}}(t)$$

Solving we obtain

$$m_{\mathrm{B}}(t) = m_{\mathrm{B}}(t_0)\,\exp\left[\frac{t}{\tau}\right] \tag{5}$$

where $\tau = \left[\{c^2 M_\odot/(1.27\times 10^{38})\}\{\epsilon/(1-\epsilon)\}\right] \sim 0.45\{\epsilon/(1-\epsilon)\}\,\mathrm{Gyr}$. If $m_{\mathrm{B}}(t) = 10^9$, $t = 0.77\,\mathrm{Gyr}$ and assuming $\epsilon \simeq 0.1$, we get the seed mass $\approx 200 M_\odot$.

Problem 7

Circular orbits around a non-rotating black hole: The gravitational potential around a Schwarzschild (non-rotating) black hole can be approximated as $\phi = -GM/(r - r_{\rm S})$, where $r_{\rm S} = 2GM/c^2$ is the Schwarzschild radius. Consider a particle with a conserved specific angular momentum l in this gravitational potential, and sketch the effective potential $\phi_{\rm eff} = \phi + l^2/(2r^2))$. For a large l, show that there is one minimum (this corresponds to the usual circular orbit in Newtonian gravity) and one maximum in the effective potential. For a small enough l, these maxima and minima merge and this orbit corresponds to the innermost stable circular orbit (ISCO). Calculate $l_{\rm ISCO}$ and $r_{\rm ISCO}$ (radius of the circular orbit) corresponding to the ISCO. Also calculate the minimum radius and specific angular momentum of a marginally bound orbit (which has zero total energy and for which the effective potential maximum just grazes zero).

Consider a particle is orbiting around a black hole of mass M. The total energy per unit mass is

$$E = {\rm KE} + {\rm PE} = \frac{1}{2}v_{\rm R}^2 + \frac{1}{2}v_{\phi}^2 - \frac{GM}{r - r_{\rm S}}$$

Since angular momentum $l = rv_{\phi}^2$ is conserved , the above equation can be written as

$$E = \frac{1}{2}v_{\rm R}^2 + \frac{l^2}{2r^2} - \frac{GM}{r - r_{\rm S}} = \frac{1}{2}v_{\rm R}^2 + \phi_{\rm eff} \quad (6)$$

where $\phi_{\rm eff} = \frac{l^2}{2r^2} - \frac{GM}{r - r_{\rm S}}$ is the effective gravitational potential.

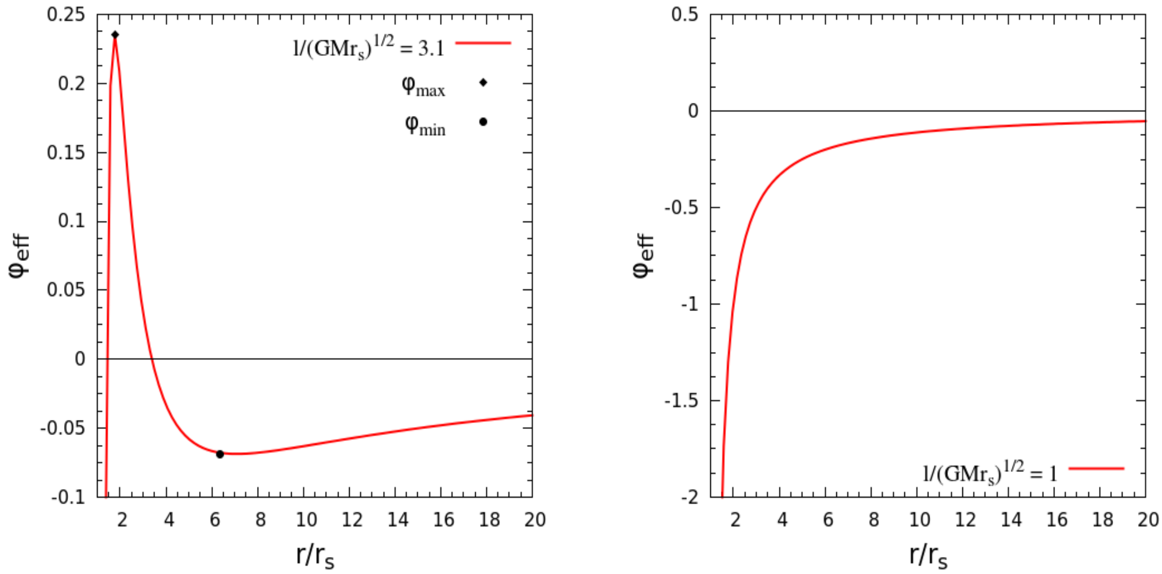

Figure 3: $\phi_{\rm eff}$ as a function of distance. Here the distance r is scaled in the unit of $r_{\rm s}$ and $\phi_{\rm eff}$ is scaled by $GM/r_{\rm s}$.

ISCO : We know that the effective potential in this problem is

$$\phi_{\rm eff} = \frac{l^2}{2r^2} - \frac{GM}{r - r_{\rm S}}$$

For inner most stable orbit, we have $d\phi_{\rm eff}/dr = 0$ and $d^2\phi_{\rm eff}/dr^2 = 0$. From $d\phi_{\rm eff}/dr = 0$, we get $l^2 = GMr^3/(r - r_{\rm s})^2$, using this and $d^2\phi_{\rm eff}/dr^2 = 0$ we find $r_{\rm ISCO} = 3r_{\rm s}$ and $l_{\rm ISCO} = (3/2)\sqrt{G\,M\,r_{\rm s}}$.

Marginally bound orbit For marginally bound orbit ($E = 0$), $dr/dt = 0$ and $d^2r/dt^2 = 0$. From the total energy equation we have

$$\frac{dr}{dt} = \sqrt{2E - \frac{l^2}{r^2} + \frac{2GM}{(r - r_s)}} = 0$$

Using $E = 0$, we get $l^2 = 2GMr^2/(r - r_s)$. Putting this in $d^2r/dt^2 = 0$ we obtain $r = 2r_s$ and $l = \sqrt{8\,G\,M\,r_s}$.

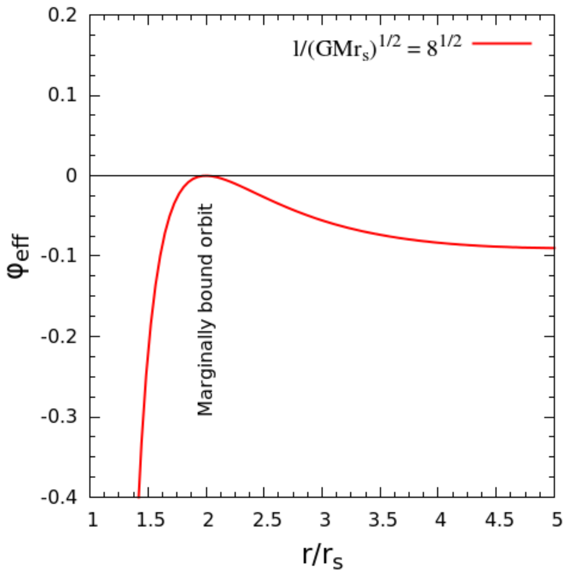

Figure 4: $\phi_{\rm eff}$ as a function of distance for Marginally bound orbit.

Problem 8 : A. R. Choudhuri/Astrophysics for physicists/Ch. 5

1. **(Problem 5.8) The Crab pulsar has period $P = 0.033$ s and characteristic slowing time $P/\dot{P} = 2.5 \times 10^3$ yr. Estimate the energy loss rate and the magnetic field by using (5.35)**

   The pulsars are believed to be powered by the rotational kinetic. The rotational energy of a spinning star is $E_{\rm rot} = (1/2)I\Omega^2$. Therefore, the energy loss rate :

$$L = \frac{dE_{\rm rot}}{dt} = I\Omega\dot{\Omega}\,, \tag{7}$$

   where the moment of inertia I can be assumed as $(2/5)MR^2$. In this problem, $P = 0.033\ {\rm s}^{-2}$ gives $\Omega = 190.4\,{\rm rad\,s^{-1}}$ and $P/\dot{P} = 2.5 \times 10^3\,{\rm yr}$ gives $\dot{\Omega} = 8.8 \times 10^{-7}\,{\rm rad\,s^{-2}}$. Assuming the mass and radius of the crab is 1.4 $M_\odot$ and $\sim 10\,{\rm km}$, we get $I = 1.12 \times 10^{45}\,{\rm g\,cm^2}$. Using all these values equation 7 yields $L = 1.9 \times 10^{41}\,{\rm erg\,s^{-1}}$.

   Considering pulsar as the spinning magnetic dipole (and assuming $\theta \approx 90^0$ in equation (5.35)), the energy loss rate is

$$L = \frac{1}{6}\frac{B_{\rm p}^2\Omega^4R^6}{c^3}$$

   From the above equation we have

$$B_{\rm p} = \sqrt{\frac{6\,L\,c^3}{\Omega^4R^6}} \tag{8}$$

   Putting all numerical values in the RHS, we get $B_{\rm p} = 1.5 \times 10^{14}\,{\rm gauss}$.

Problem 9 : Dan Maoz/Astrophysics in a Nutshell/Ch. 4

1. **(Problem 4.1) In a fully degenerate gas, all the particles have energies lower than the Fermi energy. For such a gas we found the relation between the density N_e and the Fermi momentum p_f :**

$$n_e = \frac{8\pi}{3h^3} p_f^3$$

a. For a non-relativistic electron gas, use the relation $p_f = \sqrt{2m_e E_f}$ between the Fermi momentum, the electron mass m_e, and the Fermi energy E_f in terms of n_e and m_e.
b. Estimate a characteristic n_e under typical conditions inside a white dwarf. Using the result of (a), and assuming a temperature $t = 10^7$ K, evaluate numerically the ratio E_{th}/E_f, where E_{th} is the characteristic thermal energy if an electron in a gas of temperature T, to see that the electrons inside a white dwarf are indeed degenerate.

Substituting the expression for Fermi momentum p_f in the expression for electron density n_e gives

$$E_f = \left(\frac{9n_e^2 h^6}{512\pi^2 m_e^3} \right)^{1/3}$$

For a typical white dwarf of radius 10^4 km and mass 1.4 $M_\odot$, the mean electron density (for a gas with Z/A $\sim$ 0.5) will be

$$n_e = \frac{\rho}{m_p} = 0.5 \frac{1.4M_\odot}{4/3\pi R^3 m_p} = 0.2 \times 10^{30} cm^{-3}$$

This gives a value of the Fermi energy as $E_f \simeq 2 \times 10^{-7}$ ergs. For electrons at a temperature of 10^7 K, the thermal energy is

$$E_{th} = 3/2\, k_B T = 2 \times 10^{-9}\,\text{ergs}$$

Thus the value of $E_{th}/E_f = 10^{-2}$, which justifies the fact that electrons are indeed degenerate (as seen from Fig. 4.4 of Dan Maoz)

2. **(Problem 4.3) Most of the energy released in the collapse of a massive star to a neutron star (a core-collapse supernova) is in the form of neutrinos.**
a. If the just-formed neutron star has a mass $M = 1.4M_\odot$ and a radius $R = 10$ km, estimate the mean nucleon density, in cm^{-3}. Fin the mean free path, in cm, of a neutrino inside the neutron star, assuming the density you found and a cross section for scattering of neutrinos on neutrons of $\sigma_{\nu n} = 10^{-42}$ cm^2.
b. How many seconds does it take a typical neutrino to emerge from the neutron star in a random walk ? (*Hint* : neutrinos travel at a velocity close to c. Recall that the radial distance d covered in a random walk of N steps, each of length l, is $d = \sqrt{N} l$
c. Twelve electron anti-neutrinos from Supernova 1987A were detected by the Kamiokande neutrino detector in Japan. This experiment consisted of a tank filled with 3 kton of water, and surrounded by photomultiplier tubes. The photomultipliers detect the Cerenkov radiation produced by a recoiling positron that is emitted after a proton absorbs an antineutrino from the supernova. Estimate how many people on Earth could have perceived a flash of

light, due to the Cerenkov radiation produced by the same process, when an antineutrino from the supernova travelled through their eyeball. Assume that eyeballs are composed primarily of water, each weighs about 10g, and that the Earth's population was 5 billion in 1987.

Nucleon density in the neutron star will be

$$n = (Z/A) \times (\rho/m_p) \simeq 1 \times (1.4M_{\odot}/(4/3 * \pi * 10^{18} m_p) \simeq 10^{38}\,\mathrm{cm}^{-3}$$

The mean free path would thus be

$$l = \frac{1}{n\sigma_{\nu n}} = 10^4\mathrm{cm}$$

The total distance travelled by this neutrino in a random walk upto the neutron star surface (of radius R = 10 km , case would be

$$d = \sqrt{N}l = \sqrt{R/l}l = \sqrt{lR} = 10^5\mathrm{cm}$$

Thus the time taken for a neutrino to emerge from the neutron star would be $\tau = d/c \simeq 10^{-5}s$ i.e neutrinos escape the neutron star very easily.

The probability of a person seeing a flash of light due to the neutrinos from a supernova event can be given by

$$p = N * (1 - exp(-d_{eye} n\sigma))$$

where d_{eye} is the path travelled in the eye, n is the water density in eye and σ is the cross section of interaction and N is the total number of neutrinos incident. Given that Kamiokande (with 3kton of water) detected 12 anti-neutrinos, we can estimate this probability as follows. This gives

$$12 = N * (1 - exp(-d_{kam} n\sigma)$$

This can be written as

$$exp(-d_{kam} n\sigma) = 1 - 12/N$$

d_{kam} (the path length travelled in the Kamiokande experiment) and d_{eye} will be related as

$$d_{eye}/d_{kam} = (M_{eye}/M_{kam})^{1/3} = 1.5 \times 10^{-3}$$

Thus the probability of interaction in a single person will be

$$p = N * (1 - exp(-d_{kam} n\sigma)^{d_{eye}/d_{kam}}) = N * (1 - (1 - 12/N)^{d_{eye}/d_{kam}})$$

For a large fluence of impinging neutrinos N, we can write

$$(1 - 12/N)^{d_{eye}/d_{kam}} \simeq 1 - 12 * d_{eye}/(N * d_{kam})$$

which gives

$$p = N * (1 - 1 + 12 * d_{eye}/(N * d_{kam})) = 12 * d_{eye}/d_{kam}$$

The total number of people which would have seen the event will thus be

$$N_{seen} = N_{pop} * p = 5 \times 10^9 \times 12 \times 1.5 \times 10^{-3}$$

i.e 90 million people would have seen the event !!

3. (Problem 4.4) Type-Ia supernovae are probably thermonuclear explosions of accreting white dwarfs that have approached or reached the Chandrasekhar limit.

a. Use the virial theorem to obtain an expression for the mean pressure inside a white dwarf of mass M and radius R.

b. Use the result of (a) to estimate, to an order of magnitude, the speed of sound, $v_s = \sqrt{dP/d\rho} \sim \sqrt{P/\rho}$, inside a white dwarf. In an accreting white dwarf with a carbon core that has reached nuclear ignition temperature, a nuclear burning flame encompasses the star at the sound velocity or faster. Within how much time, in seconds, does the flame traverse the radius of the white dwarf, assuming $R = 10^4\,\mathrm{km}$, $M = 1.4 M_\odot$? Note that this sound-crossing timescale is $\sim (G\rho)^{-1/2}$, which is also the free-fall timescale (Eq. 3.15.)

c. Calculate the total energy output, in erg, of the explosion, assuming the entire mass of the white dwarf is synthesized from carbon to nickel, with a mass-to-energy conversion efficiency of 0.1%. Compare this energy to the gravitational binding energy of the white dwarf, to demonstrate that the white dwarf explodes completely, without leaving any remnant.

d. Gamma rays from the radioactive decays 56 Ni $\rightarrow$ 56 Co + $\gamma \rightarrow$ 56 Fe + γ drive most of the optical luminosity of the supernova. The atomic weights of 56 Ni and 56 Fe are 55.942135 and 55.934941, respectively. Calculate the total energy radiated in the optical range during the event. Given that the characteristic times for the two radioactive decay processes are 8.8 days and 111 days, respectively, show that the typical luminosity is $\sim 10^{10} L_\odot$.

According to the virial theorem, the mean presure P inside a white dwarf can be written as $P = -1/3\frac{E}{V}$ where V is the volume and $E = -3/5\frac{GM^2}{R}$ is the gravitational binding energy of the star. Therefore,

$$P = \frac{1}{5}\frac{GM^2}{RV} = \frac{3}{20\pi}\frac{GM^2}{R^4}$$

From (a), $P/\rho = 1/3\frac{GM}{R}$, thus,

$$v_s = \sqrt{P/\rho} = \sqrt{GM/3R}$$

The time t_s it will take the flame to traverse the star is

$$t_s \sim R/v_s \sim \sqrt{\frac{R^3}{GM}} \sim \frac{1}{\sqrt{G\rho}}$$

now substituting the values of M and R, we have $t_s \sim 2.5$s

The atomic mass per nucleon of C is 1.000000amu and that of Ni is 0.998967amu, thus $\Delta m \sim 0.001033 amu$. Thus, the energy released per nucleon when Ni is synthesised from C will be

$$\Delta E = \Delta mc^2 \sim 1.54 \times 10^{-6}\mathrm{erg/nucleon}$$

and the total number of nucleons in a Ni white dwarf of mass $1.4M_\odot$ will be $1.4 \times 2 \times 10^{33} \times 6.022 \times 10^{23} = 1.68 \times 10^{57}$nucleons. Thus the total energy E_t of the explosion considering that the conversion efficiency is 0.1% will be;

$$E_t = 1.54 \times 10^{-6} \times 1.68 \times 10^{57} \times 0.001 = 2.6 \times 10^{49}\mathrm{erg}$$

The gravitational binding energy E_g of a white dwarf is given by

$$E_g = 3/5\frac{GM^2}{R} \sim 3.1 \times 10^{50}\text{erg}$$

In this case, for 56 Ni the atomic mass per nucleon is 0.998967 and for 56 Fe, it is 0.998838 so that; $\Delta m = (0.998967 - 0.998838)amu = 0.000129amu$, the change in energy per nucleon during this conversion is;

$$\Delta E = \Delta mc^2 \sim 1.9 \times 10^{-7}\text{erg/nucleon}$$

Since the gamma rays emitted during this process drives the optical luminosity observed, the total energy radiated in the optical during this process will be;

$$E_t \sim 1.9 \times 10^{-7} \times 6.022 \times 10^{23} \times 1.4 \times 2 \times 10^{33} \sim 3.2 \times 10^{50}\text{erg}$$

The characteristic time for the total reaction process takes $t = 8.8 + 111 = 119.8$days. Thus the characteristic luminosity will be;

$$L = \frac{E_t}{t} = 3.1 \times 10^{43}ergs^{-1} = 8.1 \times 10^9 L_\odot$$

4. **(Problem 4.6) A type-Ia supernova is thought to be the thermonuclear explosion of an accreting white dwarf that goes over the Chandrasekhar limit (see Problem 4.4). An alternative scenario, however, is that supernova-Ia progenitors are white dwarf binaries that lose orbital energy to gravitational waves (see Problem 5) until they merge, and thus exceed the Chandrasekhar mass and explode.**
a. Show that the orbital kinetic energy of an equal-mass binary with separation a and individual masses M is

$$E_k = \frac{GM^2}{2a}$$

and the total orbital energy (kinetic plus gravitational) is minus this amount.
b. The power lost to gravitational radiation by such a system is

$$\dot{E}_{gw} = -\frac{2c^5}{5G}\left(\frac{2GM}{c^2a}\right)^5$$

By equating to the time derivative of the total energy found in (a), obtain a differential equation for a(t), and solve it.
c. What is the maximum initial separation that a white-dwarf binary can have, if the components are to merge within 10 Gyr? Assume the white dwarfs have $1M_\odot$ each, and the merger occurs when $a = 0$.

a. n the binary system, the centrifugal force on one of the masses will be balanced by the gravitational attraction from its companion i.e

$$\frac{GM^2}{a^2} = \frac{Mv^2}{a/2}$$

therefore, the kinetic energy of one of the stars in the binary is

$$E_{k1} = \frac{mv^2}{2} = \frac{GM^2}{4a}$$

where E_{k1}=E_{k2} and the total kinettic energy is

$$E_k = 2E_{k1} = \frac{GM^2}{2a}$$

b. The time derivative of the above equation is

$$\dot{E}_k = \frac{GM^2}{2}\frac{d}{dt}(a^{-1}) = -\frac{GM^2}{2a^2}\frac{da}{dt}$$

equating $\dot{E}_k$ with $\dot{E}_{gw}$ gives

$$\frac{GM^2}{2a^2}\frac{da}{dt} = \frac{2c^5}{5G}(\frac{2GM}{c^2a})^5$$

proper evaluation of the above equation gives;

$$a^3\frac{da}{dt} = \beta$$

where $\beta = \frac{4c^5}{5G^2M^2}(\frac{2GM}{c^2})^5$, integrating the above equation will give

$$a^4 = 4\beta t$$

c. For $t = 10Gyr = 3.15 \times 10^{17}s$ and $M = 1M_\odot$, $\beta = 2.5 \times 10^{27}\,\mathrm{cm}^4\mathrm{s}^{-1}$, the maximum separation will be

$$a = (4\beta t)^{-1/4} = 2.4 \times 10^{11}\,\mathrm{cm} = 0.016\,\mathrm{AU}$$

5. **(Problem 4.7) A star of mass m and radius r approaches a black hole of mass M to within a distance $d \gg r$. a. Using Eq. 4.127, express, in terms of m, r, and M , the distance d at which the Newtonian radial tidal force exerted by the black hole on the star equals the gravitational binding force of the star, and hence the star will be torn apart.**
b. Find the black-hole mass M above which the tidal disruption distance, d, is smaller than the Schwarzschild radius of the black hole, and evaluate it for a star with $m = M_\odot$ and $r = r_\odot$. Black holes with masses above this value can swallow Sun-like stars whole, without first tidally shredding them.
c. Derive a Newtonian expression for the tangential tidal force exerted inward on the star, in terms of m, r, M , and d, again under the approximation $r \ll d$. The combined effects of the radial tidal force in (a) and and the tangential tidal force in (c) will lead to spaghettification of stars, or other objects that approach the black hole to within the disruption distance.

a. The Newtonian tidal force exerted on the star by the black hole is $F_{tide} \sim \frac{GMm2r}{d^3}$ while the gravitational binding force of the star itself is $F_g = 3/5GM^2\frac{d}{dr}(\frac{1}{r})$, when these two forces are equal;

$$\frac{GMm2r}{d^3} = 3/5GM^2\frac{d}{dr}(\frac{1}{r})$$

making d the subject of formula, we have;

$$d = (\frac{10MR^3}{3m})^{1/3}$$

b. In this case, $d = r_s = \frac{2GM}{c^2}$ when substituted into the above equation, in terms of M, the equation becomes;

$$M = \sqrt{\frac{10c^6r^3}{8G^3m}} \sim 10^8 M_\odot$$

for $m = 1M_\odot$ and $r = 1r_\odot$
c. *With respect to this problem, please refer to the first part of Problem 5 in HW1*

6. **(Problem 4.9) A spinning neutron star of mass $M = 1.4M_\odot$, constant density, and radius R=10km has a period P=1s. The neutron star is accreting mass from a binary companion through an accretion disk, at a rate of $\dot{M} = 10^{-9}M_\odot yr^1$. Assume the accreted matter is in a circular Keplerian orbit around the neutron star until just before it hits the surface, and once it does then all of the matters angular momentum is transferred onto the neutron star.**
a. Derive a differential equation for $\dot{P}$, the rate at which the neutron-star period decreases.
b. Solve the equation to find how long will it take to reach P = 1 ms, the maximal spin rate of a neutron star.

a. The Keplerian velocity of the material just before hitting the neutron star's surface is $v = (\frac{GM}{R})^{1/2}$, the angular momentum of the material per unit mass can be written as

$$\frac{J}{m} = rv = (GMR)^{1/2}$$

The rate of change of a star's angular momentum is the rate at which it receives angular momentum from the accreting matter given as;

$$\frac{d}{dt}(I\omega) = \dot{M}\frac{J}{m}$$

where the constant $I = 2/5MR^2$ and $\frac{d\omega}{dt} = -\frac{2\pi}{P^2}\frac{dP}{dt}$ since $\omega = \frac{2\pi}{P}$, substituting into the above equation, we have

$$\dot{P} = -\frac{5P^2\dot{M}(GMR)^{1/2}}{4\pi MR^2}$$

b. Rearranging the above equation gives,

$$\frac{dP}{P^2} = -\frac{5\dot{M}(GMR)^{1/2}}{4\pi MR^2}dt$$

Integrating the equation from $P = 1s$ to $P = 1ms$ and $t = 0$ to $t = t$, we have

$$t = \frac{999\times 4\pi MR^2}{5\dot{M}(GMR)^{1/2}}$$

Substituting all the values gives t.

7. **(Problem 4.10) A compact accreting object of mass M is radiating at the Eddington luminosity corresponding to that mass,**

$$L_E = \frac{4\pi cGMm_p}{\sigma_T} = 1.3\times 10^{38} ergs^{-1}\frac{M}{M_\odot}$$

An astronaut wearing a white space suit is placed at rest at an arbitrary distance from the compact object. Assuming that the cross-sectional area of the astronauts body is A = 1.5 m^2 , find the maximum allowed mass m of the astronaut, in kg, if the radiation pressure is to support her from falling onto the compact object.

The radiation pressure P_{rad} at the distance R from the compact object will be given by

$$P_{rad} = \frac{L/c}{4\pi R^2}$$

For an astronaut at this distance having an effective area $2A$, the force she experiences due to the radiation pressure will be

$$F_{rad} = \frac{2AL/c}{4\pi R^2}$$

At Eddington luminosity L_E, the force due to radiation on the Astronaut exactly balances the gravitational force she experiences due to the compact object and thus

$$\frac{2AL_E/c}{4\pi R^2} = \frac{GMm}{R^2}$$

where m is the mass of the Astronaut and M is the mass of the compact object. After substituting for L_E and rearranging, we have,

$$m = \frac{2AL_E}{4\pi G} = \frac{2\times1.5\times10^4\times1.3\times10^{38}}{4\pi\times3\times10^{10}\times6.7\times10^{-8}\times2\times10^{33}} = 7.7\times10^4\,\mathrm{g} = 77\,\mathrm{kg}$$

To compare m_p to M and σ_T to $2A$, we substitute these values into the equation for L_E. This gives

$$L_E = \frac{4\pi\times3\times10^{10}\times7.7\times10^4\times2\times10^{33}M/M_\odot}{2\times1.5\times10^4} = 1.3\times10^{38}\frac{M}{M_\odot}\,\mathrm{ergs}^{-1}$$

This reveals that L_E does not appreciably change in principle and depends on the mass M of the compact object.