IISc, Fundamentals of Astrophysics, PH 217

Solution of Problem Set 5

December 1, 2016

Problem 1

Motion of the ionization front: Assuming that a star turns ON at t = 0 in a neutral medium with H at rest, calculate the radius of the ionization front as a function of time. The volume of the ionization front increases because photoionization eats more and more into the neutral medium, as compared to recombinations in the ionized medium. Express $r_{\mathrm{I}}(t)$ in terms of the Stromgren radius and the recombination time in the ambient density plasma. Note that $r_{\mathrm{I}}(t)$ should asymptotically equal the Stromgren radius.

Consider the effective number of incident photons on the ionization front in time dt is $F_{\mathrm{eff}}dt$. These photons push the ionization front from R(t) to R(t+dt)=R+dR. Therefore, $F_{\mathrm{eff}}dt = n_0 dR$ where n_0 is the hydrogen number density of the ambient medium and F_{eff}=(incident photon - loss of photon due to the recombination)/$4\pi R^2 = \frac{S_* - \left(\frac{4}{3}\pi R^3 \dot{N}_R\right)}{4\pi R^2} = \frac{S_*}{4\pi R^2} - \frac{1}{3}Rn_0^2\beta$. Therefore the ionisation front velocity is

$$\frac{dR}{dt} = \frac{S_*}{4\pi n_0 R^2} - \frac{1}{3}Rn_0\beta(T_{\mathrm{e}}) \tag{1}$$

To find the solution, let us chose $R \to R = R_{\mathrm{s}}\lambda$ and $t = t_{\mathrm{R}}\tau$, where $R_{\mathrm{s}} = \left(\frac{3S_*}{4\pi n^2\beta(T_{\mathrm{e}})}\right)^{1/3}$ is the Stromgren radius and $t_{\mathrm{R}} = \frac{1}{n_0\beta}$. Thus the dimensionless form of above equation is

$$\frac{d\lambda}{d\tau} = \frac{1}{3}\left[\frac{1}{\lambda^2} - \lambda\right] \tag{2}$$

To solve this we can use the initial condition as $\lambda(\tau \to 0) = 0$ which gives

$$\lambda = \left(1 - e^{-\tau}\right)^{1/3} \tag{3}$$

Equation 3 shows that if $\tau >> 1$ then $\lambda \to 1$ i.e., the ionization front approaches the Strongren sphere.

Problem 2

Recall that Kapteyn deduced that the sun was located at the center of our Galaxy (in fact it was thought that ours was the only galaxy in the universe) by looking at the almost uniform distribution of stars. This was of course because dust limited

our view. Shapley, by looking at the distribution of globular clusters, deduced that there was an excess density towards the constellation of Sagittarius. Why do you think that Shapley's observations were not affected by dust attenuation?

The reason is simply because many of the globular clusters that Shapley studied are out of the dusty disc plane of the Galaxy (i.e not loacted in the galactic mid-plane) as opposed to Kapteyn's approach and so the length scales he estimated were not as severely affected by the lack of appropriate correction for the effect of dust absorption. *Ref: ned.ipac.caltech.edu/level5/ESSAYS/Cudworth.html*

Problem 3

While cosmic rays with energies less than 10^{15} eV are expected to be accelerated in Galactic supernova remnants. The ultra-high energy cosmic rays (UHECR) sources are essentially unconstrained. A powerful constrain on the plausible sources comes from the Hillas criterion which says that the Larmor radius of the UHECR should be smaller than the size of the system in order for the particle to be confined and accelerated to the relevant energy. Interpret Fig. 1 according to Hillas criterion ?

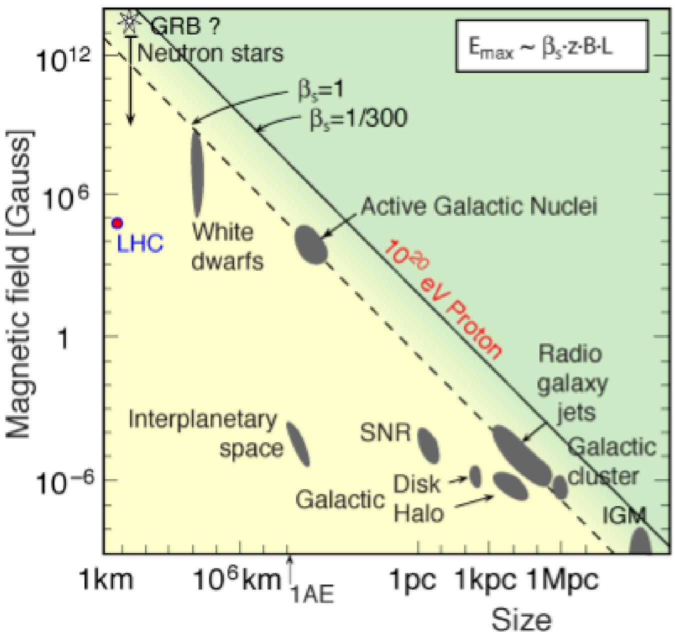

Figure 1:

The Larmor radius for relativistic particles of energy E is given roughly as $L \propto E/(qB)$, where $q = ze$ is the charge of the particle. Thus, the energy to which a particle can be accelerated is constrained by the magnetic field B and the size L of the system, as per the Hillas criterion. This leads to the expression for E_{max} as given in Fig. 1. The figure shows the extent of magnetic fields and sizes of different objects in the Universe. Diagonally running lines from top left to bottom right are lines of constant E_{max}. It is seen from the plot that no objects in the plot lie to the right of the line marked 10^{20} eV. Thus the highest energy protons which can possibly be generated (if the Hillas criterion is satisfied) is less than 10^{20} eV. Additionally the sites of the highest acceleration as seen from the plot are compact Neutron stars and GRBs. The plots are made for two values of β_s (velocity of the accelerated particles). Very large scale objects with low magnetic fields like the IGM can also possibly accelerate the particles to such high energies. However, this will increase the number of successive passes through such a medium that the

particle has to make without losing energy and thereby also increase the time scale required to accelerate these particles.
A very nice introduction to ultra high energy cosmic rays (UHECR) can be found in the review paper by Torres and Anchordoqui which can be seen at `https://ned.ipac.caltech.edu/level5/March04/Torres/Torres_contents.html`

Problem 4 : Dan Maoz/Astrophysics in a Nutshell/

1. **(Ch.5/ Problem 5.1) The oceans on Earth have a mean depth of 3.7 km and cover 71% of the Earths surface. It has been suggested that this water was brought to Earth by comets (which are composed mainly of frozen water and CO_2).**
   a. Calculate the kinetic energy of a spherical comet of radius 4 km, composed of water ice, which arrives from far away to the region of the Earth's orbit around the sun.
   b. Estimate the radius of the cylindrically shaped crater that such a comet creates when it strikes the Moon. Assume that the crater, of depth 10 km, is formed by heating to 3500 K, and thus vaporizing, a cylindrical volume of moon rocks. Moon rocks are made of silicates, which have molecular weights around 30 (i.e., a typical molecule has 30 times the mass of a hydro- gen atom), and mean solid densities $\rho \sim 2gcm^{-3}$. Ignore the latent heat required to melt and vaporize the rocks, and the energy involved in vaporizing the comet itself.
   c. The number of craters per unit area in the relatively smooth mare regions of the Moon, which trace the impact history over the past $\sim 3Gyr$, indicate a total of about 10 impacts, leaving 50-km-radius craters, during this period. Based on the assumptions in (b), these would be impacts of objects with radii $>$ 4 km. From geometrical considerations alone (i.e., the relative target sizes posed by the Earth and by the Moon, and ignoring gravity) estimate how many such objects have struck the Earth, and what is the mean time interval between impacts. How does the interval you found compare to $\sim 60Myr$, the typical interval between large extinctions of species on Earth? (The most recent large extinction, 65 Myr ago, eliminated the dinosaurs, and marked the rise of the mammals.)
   d. Assume that comets have a mass distribution $dN/dm \propto m^{-3}$, with radii ranging from 0.2 to 4 km. Based on the number of 4 km comet impacts, show that the total comet mass, if composed mainly of frozen water, is sufficient to make Earths oceans.

   a. Since the comet mass m_c moves in the Earth's orbit around the Sun, the kinetic energy E_k on the comet is provided by its gravitational potential energy given as it moves in the Sun-Earth orbit given by

   $$E_k = \frac{GM_\odot m_c}{R}$$

   where $R = 1AU$, m_c= density of ice-water x volume of comet $\sim 1 \times 4/3\pi r^3$ where $r = 4km$. Therefore,

   $$E_k = \frac{6.67\times10^{-8}\times2\times10^{33}3\times2.7\times10^{17}}{1.5\times10^{13}} \sim 2.4 \times 10^{30} erg$$

   b. On the moon, this kinetic energy is used in heating the silicate surface of the moon to temperature $\delta T = 3500K$ thereby producing the crater. Thus;

$$E_k = mc\delta T$$

where $c \sim 6.8 \times 10^6 ergg^{-1}K^{-1}$ and the mass m of pulverised silicate dust can be written as

$$m = \frac{E_k}{c\delta T} = 1.01 \times 10^{20} g$$

Since the density ρ of the silicate of the moon is $2gcm^{-3}$, the volume V of the cylindrical crater created will be

$$V = \frac{m}{\rho} = \pi r^2 h = 5 \times 10^{19}$$

where r represents the radius of the crater and the height of the crater is given as $h = 10km = 10^6 cm$ and thus,

$$r = (\frac{5 \times 10^{19}}{\pi \times 10^6})^{0.5} = 4 \times 10^6 cm = 40km$$

c. From geometrical consideration alone, the number of these objects that have impacted the Earth will be

$$10 \times \frac{R_{earth}{}^2}{R_{moon}{}^2} = 10 \times \frac{6400^2}{1737^2} \sim 140$$

The mean interval between impacts will be;

$$\frac{3Gyr}{140} \sim 21Myr$$

This reveals that impacts is about 3 times more frequent when the solar system first formed.

d. The mass distribution equation can be written as $\int dN = k \int m^{-3} dm$ where k is a constant. To evaluate k for comets with r>4km on the Earth surface, we write the expression as;

$$N = 140 = k \int_{m(4)}^{\infty} m^{-3} dm$$

$m(4) = 2.7 \times 10^{17} g$, the mass of a spherical comet with r=4km. Evaluating for k in the above equation will give;

$$k = 140 \times 2(2.7 \times 10^{17})^2 = 2.1 \times 10^{37}$$

The total mass M of the comets can be written as $\int mdN$, thus we can have;

$$M = \int_{m(4)}^{\infty} mdN = k \int_{m(4)}^{\infty} m \times m^{-3} dm = k \int_{m(4)}^{\infty} m^{-2} dm \sim 5 \times 10^{19} g$$

Surface area of portion of Earth covered by water is 71% of $4\pi R_{earth}^2$. With a depth of 3.7km, the volume occupied by water will be

$$\text{surface area x height} = 0.71 \times 4\pi (6.4 \times 10^6)^2 \times 3.7 \times 10^3 = 1.33 \times 10^{18} m^3$$

mass of water on Earth = density x volume $= 1.33 \times 10^{21} kg = 1.33 \times 10^{24}\, g$.

2. (Ch. 5/ Problem 5.2) Consider a newly formed globular cluster, with a total mass $10^6 M_\odot$, and an initial mass function $dN/dm = am^{-2.35}$ in the mass range $0.1 - 20M_\odot$, where $m \equiv M/M_\odot$.
a. Find the constant a.
b. Find the total luminosity of the cluster, assuming that all its stars are on the main sequence, and a mass-luminosity relation $L \sim M^4$. What fraction of the luminosity is contributed by stars more massive than $5M_\odot$?
c. Find the mean mass of a star in the cluster.
d. Assume that the main sequence lifetime of a $1M_\odot$ star is 10 Gyr, and main sequence lifetime scales with mass as M^{-2} . What is the mass of the most massive main-sequence stars in the cluster after 1 Gyr? What is the total luminosity of the cluster at that time?

a. The above initial mass function can be written as

$$dN = am^{-2.35}dm$$

multiplying through by m gives $mdN = am^{-1.35}dm$. Integrating both sides of the equation gives;

$$\int mdN = a\int_{0.1}^{20} m^{-1.35}dm$$

It should be noted from the above equation that $\int mdN = M$ the total mass of the star (i.e $10^6 M_\odot$). Evaluating the above integral gives;

$$a = \frac{10^6}{5.39} = 1.9 \times 10^5$$

b. Since $L \propto m^4$, then $\frac{L}{L_\odot} = m^4$ and the total luminosity can be written as;

$$L_{tot} = \int LdN = \int_{0.1}^{20} m^4 L\odot m^{-2.35}dm = L\odot \int_{0.1}^{20} m^{1.65}dm$$

Evaluating the above integral and substituting the constant a will give

$$L_{tot} = 2.0 \times 10^8 L_\odot$$

The fraction of the luminosity that will be contributed by stars more massive than $5M_\odot$ f will be given by;

$$f = \frac{aL_\odot \int_5^{20} m^{1.65}dm}{aL_\odot \int_{0.1}^{20} m^{1.65}dm} = 0.98$$

c. The mean mass of a star in the cluster M_{av} will be given by

$$M_{av} = \frac{M_{tot}}{N}$$

and this will be expressed as;

$$M_{av} = \frac{a\int_{0.1}^{20} m^{-1.35}dm}{a\int_{0.1}^{20} m^{-2.35}dm} = 0.33M_\odot$$

d. For two main sequence stars of masses m_1 and m_2 with lifetimes t_1 and t_2, in accordance with the relation $t \propto M^{-2}$ we can write;

$$t_1 m_1^2 = t_2 m_2^2$$

so that after time $t_2 = 1Gyr$ we will have

$$m_2 = \sqrt{10} = 3.2 M_\odot$$

Where $m_1 = 1M_\odot$ and $t_1 = 10Gyr$. This implies that stars with $M > 3.2M_\odot$ will no longer remain in the cluster after $1Gyr$.

At this time, the total luminosity of the cluster L_{tot} will be;

$$L_{tot} = aL_\odot \int_{0.1}^{3.2} m^{1.65} dm = 1.5 \times 10^6 L\odot$$

3. (Ch. 5/ Problem 5.3) Assume that the Milky Way, the galaxy in which we live, is composed of $5 \times 10^{10} M_\odot$ of gas, and $\sim 10^{11}$ stars, which were formed with an initial mass function $dN/dM \propto M^{-2.35}$ in the range $0.4 - 100M_\odot$.
a. What fraction of the stars is formed with a mass above $8M_\odot$, the lower limit for eventual core collapse? How many neutron stars and black holes are there in the galaxy, and roughly how much mass is there in these remnants?
b. Assume that every stellar core collapse, and the supernova explosion that follows it, distribute $0.05M_\odot$ of iron into the interstellar medium. What is the mean interstellar mass abundance of iron in the Galaxy? Compare your answer to the measured mass abundance of iron in the Sun, $Z_{Fe} = 0.00177$, and explain how this shows that the Sun is a second generation star, that was formed from pre-enriched interstellar material.
c. Several systems of binary pulsars are known, consisting of two neutron stars in close orbits. If half of all stars are in binaries, and members of binaries are formed by a random draw from the initial mass function (i.e., $P(m) \propto m^{-2.35}$), then how many pairs of stars in the Milky Way were formed in which both companions were more massive than $8M_\odot$?
d. Due to asymmetries in the supernova explosion, neutron stars are born with a kick that gives them a typical velocity of $500kms^{-1}$. What is the maximal initial separation that will allow a binary to remain bound?
e. If binaries form with an initial separation distributed uniformly between 0 and 0.01 pc, how many neutron stars binaries have survived the formation kick?

a. Following the same approach as the above equation, the fraction f of stars with masses above $8M_\odot$ will be;

$$f = \frac{\int_8^{100} M^{-2.35} dM}{\int_{0.4}^{100} M^{-2.35} dM} \sim 0.017$$

Since both black holes and neutron stars are formed from core collapse, we have

$$\text{Total no of BHs and NSs} = f \times \text{ total no of stars} = 1.7 \times 10^9$$

Total mass of BHs and NSs will be $1.7 \times 10^9 \times 1.4M_\odot \sim 2.4 \times 10^9 M_\odot$.

b. Mean interstellar mass abundance$=1.7 \times 10^9 \times 0.05M_\odot = 8.5 \times 10^7 M_\odot$. Since this value is a fraction ~ 0.0017 of the total mass of the gas in the Galaxy and since it is about same as the fraction of Iron in the Sun, it supports the the argument that the Sun must be a second generation star.

c . From the above question, the fraction f_b of binaries with initial masses of $8M_\odot$ will be;

$$f_b = \frac{\int_8^{100} P(m)m^{-2.35}dm}{\int_{0.4}^{100} P(m)m^{-2.35}dm} = \frac{8^{-3.7} - 100^{-3.7}}{0.4^{-3.7} - 100^{-3.7}} = 1.535 \times 10^{-5}$$

Since half of all the stars are estimated to be in binaries, then the number of pairs N_p formed with initial mass greater than $8M_\odot$ will be the fraction of binaries with initial masses above $8M_\odot$ multiplied by the number of binaries in the Galaxy i.e;

$$N_p = 1.535 \times 10^{-5} \times 0.5 \times 10^{11} \sim 7.7 \times 10^5$$

Since half of all the stars are estimated to be in binaries, then the number of pairs N_p formed with initial mass greater than $8M_\odot$ will be the fraction of binaries with initial masses above $8M_\odot$ multiplied by the number of binaries in the Galaxy i.e;

$$N_p = 1.535 \times 10^{-5} \times 0.5 \times 10^{11} \sim 7.7 \times 10^5$$

d. Equating the binding energy between two $8M_\odot$ stars at separation r to the kinetic energy of two $1.4M_\odot$, we have

$$\frac{GM^2}{r} = 2(1/2mv^2)$$

where $M = 8M_\odot$ and $m = 1.4M_\odot$, making r the subject of formula and making appropriate substitutions, we have;

$$r = \frac{GM^2}{mv^2} = 2.45 \times 10^{12} cm$$

e. The number N of neutron star binaries that have survived the formation kick will be

$$\frac{r}{0.01pc} \times 7.7 \times 10^6 = \frac{2.45 \times 10^{12}}{3.1 \times 10^{16}} \times 7.7 \times 10^6 \sim 608$$

N.B: 7.7×10^6 is used instead of 7.7×10^5 which is the value we obtained just to match the given answer in the problem, the right answer may be ~ 60

4. **(Ch.5 / Problem 5.4) A new star lights up inside a cloud of atomic hydrogen with a constant number density of n atoms per unit volume. The star emits ionizing photons at a rate of Q_* photons per unit time. The ionizing photons begin carving out a growing Stromgren sphere of ionized gas inside the neutral gas.**
a. At a distance r from the star, what is the timescale τ_{ion} over which an individual atom gets ionized, if the ionization cross section is σion?
b. If the recombination coefficient is $\alpha \equiv \langle \sigma v \rangle$, what is the timescale τ_{rec} for an individual proton to recombine with an electron?
c. At a position close to the star, where the ionizing flux is high, and therefore $\tau_{ion} \ll \tau_{rec}$, show that the velocity at which the ionization front that bounds the Stromgren sphere advances is $v_{if} = Q_*/(4\pi r^2 n)$.
d. Evaluate v_{if} for $Q_* = 3 \times 10^{49} s^{-1}$, $n = 10^4 cm^{-3}$, and for $r = 0.01$ pc, 0.05 pc , and 0.1 pc, respectively. From $v_{if}(r)$, obtain and solve a simple differential equation for $r_{strom}(t)$, and find roughly how long it takes the ionization front to reach the final Stromgren radius (0.2 pc for these parameters; see Eq. 5.27).

At the distance r, the volume of the sphere is $4/3\pi r^3$ and the number of atoms in the sphere will be $4/3\pi r^3 n_{ion}$ and therefore, the time scale of ionization is

$$\tau_{ion} = \frac{4/3\pi r^3 n_{ion}}{Q_*}$$

b. The recombination timescale is simply

$$\tau_{rec} = \frac{1}{n_{ion}\alpha}$$

c. In a fast ionisation front, the ram pressure of the gas entering the front is matched by the sum of the gas pressure and ram pressure of the ionised gas leaving the front. At a region close to the star, only a fraction of the emitted photons are used to maintain the ionisation, the remaining photons push an ionisation front through the neutral medium at a velocity $v_{if} = \frac{dr}{dt}$. If we assume that the fully ionised region within the front is in equilibrium with the neutral gas beyond it, then

$$4\pi r^2 n \frac{dr}{dt} = Q_* - \frac{4\pi}{3}\alpha(T) n^2 r^3$$

Very close to the star, ionisation will dominate the process since $\tau_{ion} \ll \tau_{rec}$ and the second part of the above equation can be ignored so that we have;

$$v_{if} = \frac{Q_*}{4\pi r^2 n}$$

d. Substituting the appropriate values of r in the equation for v_{if} gives the answers.

$$v_{if} = \frac{dr}{dt} = \frac{Q_*}{4\pi r^2 n}$$

this implies that

$$r^2 dr = \frac{Q_* dt}{4\pi n}$$

Where $r = r_{strom}(t)$. Integrating the above equation and substituting the given values in the problem, we have;

$$t = \frac{4\pi n r^3}{3Q_*} \sim 10 years$$

5. **(Ch. 6 / Problem 6.1) Even when distances to individual stars are not known, much can be learned simply by counting stars as a function of limiting flux. Suppose that, in our region of the galaxy, the number density of stars with a particular luminosity L, n(L), is independent of position. Show that the number of such stars observed to have a flux greater than some flux f_0 obeys $N(f > f_0) \propto f_0^{-3/2}$. Explain why the same behaviour will occur even if the stars have a distribution of luminosities, as long as that distribution is the same everywhere. If you observed that the numbers do not grow with decreasing f_0 according to this relation, what could be the reason?**

   The number N of stars that have luminosities between L and $L + dL$ observed can be related to the number density $n(L)$ by the equation;

$$N = \int_0^d n(L) dV$$

where d the radius of the sphere centered around the Solar system. Also, L is related to f by;

$$f = \frac{L}{4\pi d^2}$$

and so d can be expressed as

$$d = (\frac{L}{4\pi f})^{1/2}$$

and since $N \sim n(L)V$ and so for $f > f_0$ we can write N as

$$N(f > f_0) \sim n(L)[(\frac{L}{4\pi f_0})^{1/2}]^3 \sim \frac{n(L)L^{3/2}}{(4\pi)^{3/2}} f_0^{-3/2}$$

This is because, as stated above, what is observed is the total number of stars with fluxes $f > f_0$ and not the number of stars at distances smaller than a certain value and so provided the distribution of luminosity is same everywhere, the same behaviour will be observed even if the stars have a distribution of luminosities.

If the number of stars does not grow with decreasing f_0, it implies that below the limiting value of f_0 the stars are significantly deem and the condition $f > f_0$ has plausibly been violated.

6. **(Ch. 6 / Problem 6.2) Derive the expression for gravitational focusing - the increase in the effective cross section for a physical collision between two objects due to their gravitational attraction (Eq. 6.12), as follows. Consider a point mass approaching an object of mass M and radius r_0. When the distance between the two is still large, their relative velocity is v_{ran} and the impact parameter (i.e., the distance of closest passage if they were to continue in relative rectilinear motion) is b. Due to gravitational attraction, the point mass is deflected toward the object and, at closest approach, grazes the object's surface at velocity v_{max}.**
a. Invoke energy conservation to show that $v_{\text{ran}}^2 = v_{\text{max}}^2 - v_{\text{esc}}^2$, where $v_{\text{esc}} = (2GM/r_0)^{1/2}$ is the escape velocity from the surface of the star.
b. Show that angular momentum conservation means that $bv_{\text{ran}} = r_0 v_{\text{max}}$.
c. Combine the results of (a) and (b) to prove that gravitational focusing results in an effective cross section for a collision that equals the geometrical cross section of the object times a factor $(1 + v_{\text{e}}^2/v_{\text{ran}}^2)$.

a. When the test object is far away from the massive body, its total energy can be considered as completely kinetic i.e. $E_{\text{i}} \approx \text{KE} = (1/2)m\, v_{\text{ran}}^2$. As the object comes close to the massive object, its total energy becomes $E_{\text{f}} = \text{KE} + \text{PE} = (1/2)m\, v_{\text{max}}^2 - G\, M\, m/r_0^2$. From conservation of energy we have $E_{\text{i}} = E_{\text{f}}$, which gives

$$v_{\text{ran}}^2 = v_{\text{max}}^2 - v_{\text{esc}}^2$$

where $v_{\text{esc}} = \sqrt{2GM/r_0}$.

b. Since angular momentum is defined as the moment of linear momentum, it can be directly shown $bv_{\text{ran}} = r_0 v_{\text{max}}$.

c. Substituting v_{max} from (b), $v_{\text{max}} = bv_{\text{ran}}/r_0$ in (a) we get $b^2 = r_0^2 \left[1 + (v_{\text{esc}}/v_{\text{ran}})^2\right]$. Therefore, the effective cross-sectional area $\pi b^2 = A\left[1 + (v_{\text{esc}}/v_{\text{ran}})^2\right]$, where $A = \pi r_0^2$ is the geometrical cross-section of the massive object.

7. **(Problem 6.3) In the Solar neighbourhood, the Milky Way has a flat rotation curve, with $v(r) = v_{\text{c}}$, where v_{c} is a constant, implying a mass density profile**

$(r) \sim r^{-2}$ **(Eq. 6.18).**

a. Assume there is a cut-off radius R, beyond which the mass density is zero. Prove that the velocity of escape from the galaxy from any radius $r < R$ is

$$v_{\rm e}^2 = 2\, v_{\rm c}^2\, [1 + ln(R/r)]$$

b. The largest velocity measured for any star in the Solar neighbourhood, at $r = 8$ kpc, is 440 km s^{-1}. Assuming that this star is still bound to the galaxy, find a lower limit, in kiloparsecs, to the cutoff radius R, and a lower limit, in units of M , to the mass of the galaxy. The Solar rotation velocity is $v_{\rm c} = 220$ km s^{-1}.

a. To escape from a galaxy, the kinetic energy of the object $\geq$ gravitational potential energy. This yields

$$\frac{1}{2} m\, v_{\rm e}^2 = \int_r^R \frac{GM(r)}{r^2} + \int_R^\infty \frac{GM_{\rm tot}}{r^2}\ ,$$

where $M(r) = v_{\rm c}^2 r/G$ is the mass contained within the radius r and $M_{\rm tot}$ is the total mass of the galaxy (i.e., upto radius R). Integrating we obtain

$$v_{\rm e}^2 = 2\, v_{\rm c}^2\, [1 + ln(R/r)] \tag{4}$$

b. From 4, we have

$$R \geq r\ \exp\left[\frac{v_{\rm e}^2}{2\, v_{\rm c}^2} - 1\right]$$

Using $r = 8$ kpc, $v_{\rm e} = 440$ km s^{-1} and $v_{\rm c} = 220$ km s^{-1}, we get $R_{\rm min} = 21.7$ kpc. The minimum mass of the Galaxy is $M_{\rm min} = R_{\rm min} v_{\rm c}^2/G = 2.43 \times 10^{11}\, M_\odot$.

8. **(Problem 6.7) Modified Newtonian Dynamics (MoND) proposes that, for small accelerations, Newtons second Law, $F = ma$, approaches the form $F = ma^2/a_0$, where a_0 is a constant (see Eq. 6.50).**

   a. Show how such an acceleration law can lead to flat rotation curves, without the need for dark matter.

   b. Alternatively, propose a new law of gravitation to replace $F = GMm/r^2$ at distances greater than some characteristic radius r_0 , so as to produce flat rotation curves without dark matter. Make sure your modified law has the right dimensions.

   c. Modify further the gravitation law you proposed in (b) with some mathematical formulation (many different formulations are possible), so that the law is Newtonian on scales much smaller than r_0, with a continuous transition to the required behavior at $r = r_0$.

a. The gravitational force on a particle of mass m is

$$F = \frac{GMm}{r^2}$$

According to MOND, the LHS of the above equation can be written as ma^2/a_0 where $a = v^2/r$ is the centrifugal acceleration. This implies

$$\frac{m}{a_0}\left(\frac{v_{\rm c}^2}{r}\right)^2 = \frac{GMm}{r^2}$$

$$v_{\rm c} = (G\, M\, a_0)^{1/4} \tag{5}$$

Note that, for our Galaxy, $a_0 \sim 10^{-8}$ cm s^{-2}, $M \sim 10^{11}\, M_\odot$ km s^{-1} i.e., $v \sim 190$ km s^{-1} which is very close to the observed rotational velocity.

b. Let us define $F = (GMm/r^2)\, g(r)$ where $g(r)$ is defined as

$$g(r) = \begin{cases} 1, & \text{if } r << r_0 \\ r/r_0, & \text{if } r >> r_0 \end{cases} \tag{6}$$

The equation of motion is

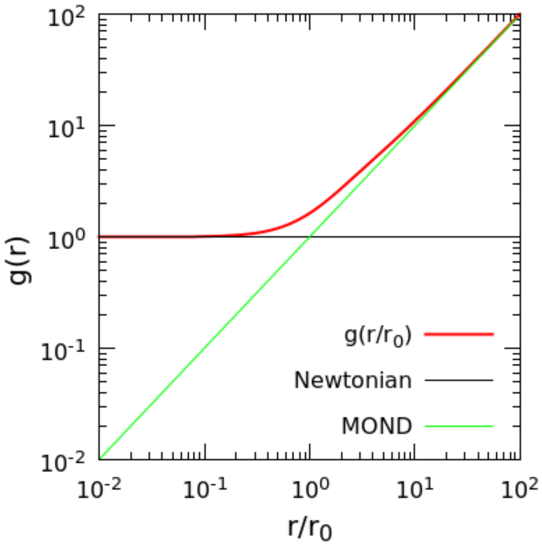

Figure 2: g(r) as a function of r/r_0.

$$\frac{v_c^2}{r} = \frac{GM}{r^2}\, g(r) \tag{7}$$

At larger distance, the above equation shows $v_c = (G\,M/r_0)^{1/2}$ i.e., the flat rotational curve. For Galactic parameters $r_0 \sim 10$ kpc, $M \sim 10^{11}\, M_\odot$ gives $v_c \sim 207$ km s^{-1}.

c. A continuous function of $g(r)$ can be chosen as

$$g(r) = \left[1 + \left(\frac{r}{r_0}\right)\left(1 - e^{-r/r_0}\right)\right]$$

Note that, this choice satisfies the condition given in equation 6, see Figure 2. It is worth mentioning that the modification done here is not unique.

Problem 5 : A. R. Choudhuri/Astrophysics for physicists

1. **(Ch. 6 / Problem 6.1) We have presented a very elementary discussion of star count analysis in section 6.1.1 by assuming that all stars have the same absolute magnitude M and there is no absorption in interstellar space. Now assume that a fraction of stars $(M)\ dM$ have absolute magnitudes between M and $M + dM$, whereas $a(r)$ is the change in magnitude of a star at a distance r due to absorption. Suppose $A(m)dm$ is the number of stars within a solid angle**

ω having apparent magnitude between m and $m + dm$. If $D(r)$ is the number density of stars at a distance r, show that

$$A(m) = \omega \int_0^\infty \Phi[m + 5 - 5\text{Log}(r) - a(r)]D(r)r^2 dr$$

Show that this expression reduces to the form equation (6.3) if all stars have the same absolute magnitude with no absorption and are uniformly distributed.

The apparent magnitude of a star is $m = m_{\text{ref}} - 2.5\text{Log}f(r) + \text{constant}$, where $f(r)$ is the flux coming from an object at a distance r. According to the definition of the absolute magnitude (M), $m = M - 2.5\text{Log}[f(r)/f(10\text{pc})] = M + 5\,\text{Log}[r/(10\text{pc})]$ which yields $M = m + 5 - 5\text{Log}[r]$. Now in the presence of dust absorption, the apparent magnitude becomes $m = m_{\text{ref}} - 2.5\text{Log}f + \text{constant} + a(r)$, where $a(r)$ represents the dust absorption. Note that the apparent magnitude increases with the increase of a. Therefore, in the presence of dust, the absolute magnitude is

$$M = m + 5 - 5\text{Log}(r) - a(r)$$

which gives

$$A(m) = \int_0^\infty \omega\, r^2\, dr\, D(r)\, \Phi[M] = \omega \int_0^\infty \Phi[m + 5 - 5\text{Log}(r) - a(r)]D(r)r^2 dr \qquad (8)$$

If all the stars are having same absolute magnitude and uniformly distributed (i.e., $D(r) = D$ and $\Phi[M] = \Phi$) in the sky, then

$$A(m)|_r = \int_0^r D\omega\, r^2\, dr = \frac{1}{3} r^3 \omega D \qquad (9)$$

Neglecting the dust absorption, we have $M = m + 5 - 5\text{Log}(r)$ i.e., $r = 10^{1+0.2[m-M]}$ pc. Using the equation of r, from equation 9, we get

$$A(m)|_r = \left(\frac{10^3}{3} \omega D 10^{-0.6M} \right) 10^{0.6m} \propto 10^{0.6m}$$

which is the required answer.

2. **(Ch. 6 / Problem 6.2) The interstellar medium in the galactic disk diminishes the luminosity of stars by about 1.5 magnitude (i.e. increases the magnitude by 1.5) per kpc. Show that this implies that the brightnesses of stars fall off with distance r in the galactic disk as**

$$\frac{e^{-\alpha r}}{r^2}$$

Find the value of α.

From problem 6.1, we have $m = M - 5 + 5\text{Log}(r) + a(r)$. The extinction factor $a(r)$ is defined as

$$a(r) = -2.5\,\text{Log}[I_{\text{obs}}/I_{\text{emt}}]\ ,$$

where $I_{\text{obs}} = I_{\text{emt}} e^{-\tau(r)}$ and $\tau(r) = \int_0^r dr\, \alpha \approx \alpha\, r$ is the optical depth. This gives

$$m = M - 5 + 5\text{Log}(r) - 2.5\text{Log}(e^{-\alpha\, r}) \equiv M - 5 - 2.5\text{Log}\left[\frac{e^{-\alpha\, r}}{r^2}\right]$$

This shows that the brightnesses of stars fall off with distance r as $\frac{e^{-\alpha r}}{r^2}$.

From the above expression, we obtain

$$\alpha = -\frac{1}{r}\ln\left[r^2\,10^{-0.4(m-M+5)}\right] = 2.4$$

3. (Ch. 6 / Problem 6.4) Make a simplified model of the galactic disk by assuming it to be an infinite sheet of constant thickness with constant density inside. Show that a star displaced from the mid-plane of the Galaxy in the vertical direction undergoes simple harmonic oscillations around the mid-plane (assuming that the star always remains within the region of constant density). Taking the density in the mid-plane to correspond to about 5×10^6 hydrogen atoms m^3, estimate the period of oscillation. How does it compare with the period of revolution of a star in the solar neighbourhood around the galactic centre?

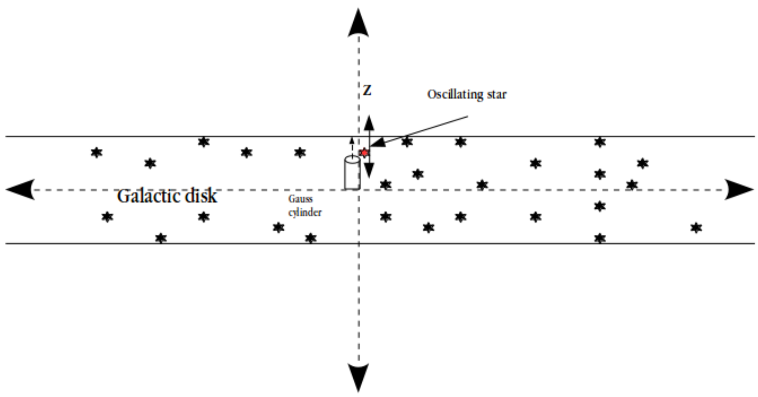

Figure 3:

The field from an infinite sheet of constant thickness can be found from applying Gauss's law to gravitational fields, which states

$$\nabla.\vec{g} = -4\pi G\rho$$

Using the divergence theorem and integrating this gives

$$\oint \vec{g}.d\vec{A} = -4\pi G \int_V \rho dV$$

For a sheet of uniform density, ρ is constant. To find the force on a star displaced at a height z from the mid-plane of the galaxy, we take a cylindrical surface over which we can apply the Gauss law (see Fig. 2). The left hand side of the integral equation will simply be equal to $\vec{g}(z)A$ (as the field flux exiting all sides except the top of the cylinder is zero). The right hand side will be equal to $-4\pi G\rho Az$. Here A is the area of the top surface of the cylinder. Thus

$$\vec{g}(z)A = -4\pi G\rho Az$$

which gives the force on the star at a height z to be equal to $M_*\vec{g}(z)$, which is

$$\vec{F}(z) = -4\pi G\rho M_* z$$

This equation is similar to the equation for a harmonic oscillator

$$F(x) = -kx$$

with the spring constant $k = 4\pi G\rho M_*$.

The period of oscillation for this oscillator will be given as

$$P = 2\pi\sqrt{\frac{M_*}{k}} = 2\pi\sqrt{\frac{1}{4\pi G\rho}}$$

On substituting the value of the constants (with $\rho = 5m_p \times 10^6$) , this period comes out to be $P \sim 0.7 \times 10^8$ yr, which is about half the value of the period that solar neighbourhood stars take to rotate about the centre of the galaxy ($P_{rot} = 2 \times 10^8$ yr - see equation 6.9 of Arnab's book).

4. **(Ch. 6 / Problem 6.6)**

   Consider an atom with three levels denoted by 1, 2 and 3 in order of increasing energy. Suppose no transitions take place between the upper two levels 2 and 3. Writing balance equations of the type (6.59) and assuming that the radiation present is not strong enough to make radiative transitions important, show that

$$\frac{n_3}{n_2} = \frac{g_3(1 + A_{21}/n_e\gamma_{21})}{g_2(1 + A_{31}/n_e\gamma_{31})}e^{-E_{23}/k_BT}$$

   Here all the symbols have obvious meanings. It is clear that we shall have the Boltzmann distribution law when n_e is large. Discuss the conditions which would lead to population inversion. If there is no transition between the upper two levels, then this population inversion may not give rise to maser action. But this simple example of a three-level system should give some idea of how population inversions can arise.

5. **(Ch. 6 / Problem 6.7)**

   We have pointed out in §6.6.3 that CO molecules in molecular clouds emit at frequencies which are integral multiples of 115 GHz. If I is the moment of inertia of the molecule around an axis perpendicular to the axis of the molecule, then show that the energy levels of the molecule are given by

$$E_J = \frac{\hbar^2}{2I}J(J+1)$$

   where J can have integral values. If the selection rule $\Delta J = 1$ has to be obeyed for emission, then show that the emission spectrum should be as seen. Make an estimate of the distance between the carbon and the oxygen atoms in the molecule.

For a di-atomic molecule rotating about an axis as shown in Fig. 3, the angular momentum of rotation is given by $L = I\omega$. Here ω is the angular velocity and

$$I = m_c r_e^2$$

is the moment of inertia about the center of mass. m_c is the reduced mass of the system and r_e is the distance between the two atoms in the molecule. The rotational energy related to this angular momentum is given as

$$E_{rot} = \frac{I\omega^2}{2} = \frac{L^2}{2I}$$

Quantization of the angular momentum $L = n\hbar$ in the atom, leads to quantization of the rotational energy levels too. These levels can be found from the eigen values of the Schroedinger equation, which gives the required relation. For selected transitions where ΔJ = -1, the transition energies, will be obtained as

$$\Delta E_{rot} = \frac{\hbar^2}{2I}\left(J(J+1) - (J-1)J\right) = J\frac{\hbar^2}{I}$$

Thus the transition energies ΔE_{rot} which are seen as the emission lines are just integral multiples of the quantity $\hbar^2/(I)$. The frequency corresponding to this is obtained by

$$h\nu = \hbar^2/(I);\ \nu = h^2/(4\pi^2 I)$$

Putting the value of I, with

$$m_c = \frac{m_O m_C}{m_O + m_C} = 16m_p/3$$

gives

$$r_e = \sqrt{\frac{3h}{64\pi^2 \nu m_p}} = 1.3A^\circ$$

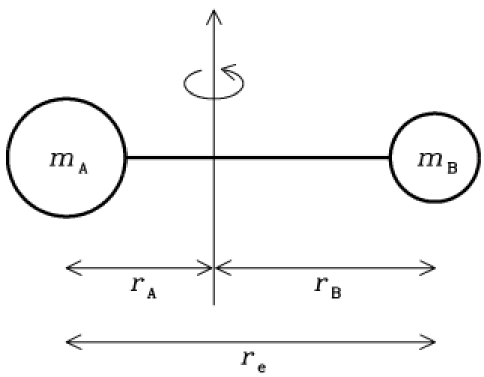

Figure 4: A diatomic molecule rotating about an axis perpendicular to axis of the molecule - Figure courtesy `http://www.cv.nrao.edu/course/astr534/MolecularSpectra.html`

6. **(Ch. 9 / Problem 9.1)**

   Suppose an elliptical galaxy appears circular in the sky with the fall in surface brightness given by the de Vaucouleurs law (9.2). Show that the total light coming from this galaxy given by $\int_0^\infty I(r)2\pi r dr$ is equal to $7.22\pi r_e^2 I_e$. Show also that the light coming from within r_e is exactly half this amount.

The total intensity I_t is given as

$$I_t = \int_0^\infty I(r)2\pi r dr = \int_0^\infty 2\pi I_e r_e \, exp\left\{-a\left((\frac{r}{r_e})^b - 1\right)\right\} dr$$

Here $a = 7.67$ and $b = 0.25$ To solve this, substitute $x = \left(\frac{r}{r_e}\right)^b$. The limits remain the same and

$$dx = b\frac{a}{r_e^b}r^{b-1}dr = b\frac{a}{r_e^b}r^{b-2}rdr$$

This can be rewritten using $r = r_e(x/a)^{1/b}$ as

$$rdr = r_e^2 a^{2/b} x^{\frac{2-b}{b}} dx$$

Substituting value of $b = 0.25$, we can write the integral as

$$I_t = 8\pi I_e a^{-8} r_e^2 e^a \int_0^\infty x^7 e^{-x} dx$$

Using Gamma function, we can write

$$\int_0^\infty x^7 e^{-x} dx = \Gamma(8) = 7!$$

This gives

$$I_t = (8a^{-8}e^a 7!)\pi I_e r_e^2 = 7.22\pi I_e r_e^2$$

For the intensity inside the radius r_e, we integrate the orginial function from 0 to r_e. This translates to integration from 0 to a in the changed variable x

$$I_{1/2} = 8\pi I_e a^{-8} r_e^2 e^a \int_0^a x^7 e^{-x} dx$$

Using the incomplete Gamma function

$$\int_0^a x^7 e^{-x} dx = \Gamma(8, a) = 7! e^{-a} \sum_{i=0}^{7} \frac{a^i}{i!}$$

Gives

$$I_{1/2} = I_t e^{-a} \sum_{i=0}^{7} \frac{a^i}{i!} \simeq \frac{I_t}{2}$$

Thus the light coming from within r_e is half the light coming from the whole galaxy

7. **(Ch. 9 / Problem 9.3) Suppose a plasma jet is coming from a quasar at a relativistic speed 0.98c. At what angle with respect to the line of sight must it lie to cause the maximum superluminal motion? What is the value of the maximum apparent transverse velocity ?**

Superluminal tranverse velocity is given as

$$v_\perp = \frac{v sin\theta}{1 - \beta cos\theta}$$

where $\beta = v/c$. Maximising this by equating its derivative to zero, we get

$$\frac{dv_\perp}{d\theta} = \frac{cos\theta}{1 - \beta cos\theta} - \frac{\beta sin^2\theta}{(1 - \beta cos\theta)^2} = 0$$

This on solving gives

$$(1 - \beta cos\theta)cos\theta = \beta sin^2\theta$$

$$cos\theta = \beta(sin^2\theta + \cos^2\theta)$$

$$\theta = cos^{-1}\beta$$

For $\beta = 0.98$, $\theta = 11.5°$ and

$$v_\perp = \frac{vsin\theta}{1 - \beta cos\theta} = c\frac{\beta}{\sqrt{1-\beta^2}} = 4.92c$$