PROPORTION TYPE DIFFERENTIAL EQUATIONS

1. Due to a curse imposed by a neighboring town, the members of a particular town are driven away. The rate at which the population is declining is approximately $2\sqrt{P}$ people per month when the population is P. Suppose the population initially has 1600 members.

A. Write an initial value problem for the population at time t.

$$\frac{dP}{dt} = -2\sqrt{P} \qquad P(0) = 1600$$

B. Solve the initial value problem.

$$\int P^{-\frac{1}{2}}\,dP = \int -2\,dt$$

$$\frac{1}{2}\sqrt{P} = -2t + C$$

$$\sqrt{P} = -4t + C$$

$$P = (-4t + C)^2$$

$$P(0) = C^2 = 1600 \quad C = 40$$

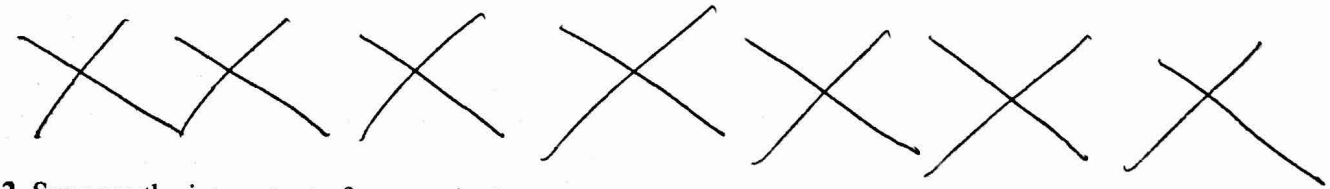
$$P(t) = (-4t + 40)^2$$

C. Use your solution to determine when everyone will have left the town.

$$0 = (-4t + 40) \qquad t = 10 \text{ months}$$

D. Plot the slope field. Are your answers to parts A and B consistent to your slope field? $0 \le t \le 50$, $0 \le P \le 1800$ is a reasonable window.

E. Would Euler's method generate an under estimate or over estimate of the actual population values? How do you know?



2. Suppose the interest rate for a particular type of bank account has been decreasing linearly as indicated in the table below. Write a differential equation satisfied by M, the amount of money in the account at time t, measured in years since 2002.

Year	2002	2003	2004
Interest rate	4.6%	4.2%	3.8%

$$.046 - .04t$$

$$\frac{dM}{dt} = (.046 - .04t)\,M$$

3. The rate at which ice forms on a lake is inversely proportional to the thickness of the ice at time t. Suppose the ice is initially 1 inch thick and initially forming at the rate of 3 inches per day.

A. Find the thickness of the ice as a function of time measured in days.

Let T be thickness

$\frac{dT}{dt} = \frac{k}{T}$ $\quad T(0) = 1$ $\quad k = 3$

$\frac{dT}{dt} = \frac{3}{T}$ $\quad \int T\,dt = \int 3\,dt$

$\frac{T^2}{2} = 3t + C$

$T^2 = 6t + C$

$T = \pm\sqrt{6t + C}$

$T(0) = \sqrt{C} = 1$

$C = 1$

$T = \sqrt{6t + 1}$

B. When will the ice be 2.5 inches thick?

$2.5 = \sqrt{6t+1}$

$6.25 = 6t + 1$

$5.25 = 6t$

$t = \frac{5.25}{6}$ days

4. This is how linguists use language to help determine the age of a civilization:

* they make a list of basic words in the original language (house, man, etc.)
* they determine how many words are still in use
* they assume the rate at which words leave the list is proportional to the number of words still in use from the list.

The basic list for classical Latin has 210 words with 144 still in use in modern Italian. Suppose the proportionality constant is $k = 0.22$. Let t be measured in thousands of years. How long did it take modern Italian to develop?

Let W be words in list.

$\frac{dW}{dt} = -kW = -.22W.$

$\int \frac{dW}{W} = \int -.22\,dt$

$\ln W = -.22t + C$

$W = Ce^{-.22t}$

$W(0) = 210 \quad C = 210$

$W = 210e^{-.22t}$

$144 = 210e^{-.22t}$

$\frac{144}{210} = e^{-.22t}$

$\frac{\ln\left(\frac{144}{210}\right)}{-.22} = t$

$t = 1.715$ years.

DIFFERENCE AND SUM TYPE DIFFERENTIAL EQUATIONS

1. Dead leaves accumulate on the ground at a rate of 4 grams per square centimeter per year. At the same time they are decomposing at a continuous rate of 64% per year.

A. Write a differential equation for the amount of dead leaves (per square centimeter) at time t.

Let D(t) be dead leaves per sq cm at time t.

$$\frac{dD}{dt} = 4 - .64D$$

B. Find the equilibrium solution. What happens to the amount of dead leaves on the ground if the initial amount is more than the equilibrium? Less than the equilibrium? Is the equilibrium solution a stable solution.

$4 - .64D = 0$

$4 = .64D$

$D = \frac{4}{.64} = 6.25$

above $4 - .64(>6.25) < 0$ ← negative

6.25

below $4 - .64(<6.25) > 0$

Stable equilibrium.

2 An investor has \$1000 to open a bank account and plans to add \$800 per year. All funds in the account will earn 5% annual interest compounded continuously.

A. Write an initial value problem for the amount of money in the account at time t.

Let M be amount of money at time t.

$$\frac{dM}{dt} = 800 + .05M \qquad M(0) = 1000$$

B. Solve the initial value problem.

$$\int \frac{dM}{800 + .05M} = \int dt$$

$$\frac{1}{.05} \cdot \ln|800 + .05M| = t + C$$

$$\ln|800 + .05M| = .05t + \tilde{C}$$

$$800 + .05M = Ce^{.05t}$$

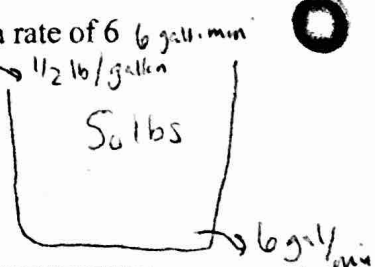
$$= \frac{Ce^{.05t} - 800}{.05}$$

$M = Ce^{.05t} - 16000$

$C = 17000$

$M = 17{,}000e^{.05t} - 16000$

3. A 120 gallon tank is full of water in which S_0 pounds of salt are dissolved. Starting at time $t = 0$ minutes, a salt solution containing 1/2 pound of salt per gallon is added to the tank at a rate of 6 gallons per minute. Assume the solution mixes instantly and any excess drains off.



A. Write an initial value problem for the amount of salt in the tank at time t.

Let S be amount of salt in tank

$$\frac{dS}{dt}\ (\text{lbs/min}) = \frac{1}{2}\ \text{lb/gal} \cdot 6\ \text{gal/min} - \frac{S\ \text{lbs}}{120\ \text{gal}} \cdot \frac{6\ \text{gal}}{1\ \text{min}}$$

$$\frac{dS}{dt} = 3 - \frac{S}{20}$$

$$\boxed{\frac{dS}{dt} = 3 - \frac{S}{20} \qquad S(0) = S_0}$$

B. Write an initial value problem for the concentration of salt in the tank at time t.

Let C be conc of salt in tank

$$\frac{dC}{dt} = \frac{\frac{1}{2}\ \text{lb/gal} \cdot 6\ \text{gal/min}}{120\ \text{gal}} - \frac{\left(\frac{S\ \text{lbs}}{120\ \text{gal}}\right) \cdot \frac{6\ \text{gal}}{\text{min}}}{120\ \text{gal}}$$

$$= \frac{3}{120} - \frac{6C}{120}$$

$$\boxed{\frac{dC}{dt} = \frac{1}{40} - \frac{C}{20}} \qquad \boxed{C(0) = \frac{S_0}{120}}$$

4. Biologists introduced a new variety of fish into a lake. Their model predicts that the population should double every 18 months. Which model are the biologists using? Assume $k > 0$.

$\frac{dA}{dt} = k$ $\qquad$ $\boxed{\frac{dA}{dt} = kA}$ $\qquad$ $\frac{dA}{dt} = k(A_0 - A)$

doubles means rate of increase is 2t

Proportional ~~equal~~ to population size

5. Pieces of equipment fail at a rate proportional to the fraction of equipment that has not failed by time t.

A. Write a differential equation for $F(t)$, the fraction of equipment failing at time t.

$$\frac{dF}{dt} = k(1 - F)$$

T not failed.

B. Find $\lim_{t \to \infty} F(t)$.

equilibrium → F = 1, can't be more than one

So stable.

below 1 positive dF/dt

$\lim_{n \to \infty} F(t) = 1$.