Homework 6 **Math 563 Fall 2016**

P and R (post and referee) problems: Solutions posted by 11/21, referee reports by 11/28.

1. Section 5.1, problem 5.1.5

2. Section 5.2, problem 5.2.1

3. Let X be an integrable random variable on the probability space $(\Omega, \mathcal{F}P)$ if $\mathcal{G}_1 \subset \mathcal{G}_2 \subset \mathcal{F}$ are sub-σ-fields, prove that

$$E(E(X|\mathcal{G}_1)|\mathcal{G}_2) = E(X|\mathcal{G}_1) \quad \text{and} \quad E(E(X|\mathcal{G}_2)|\mathcal{G}_1) = E(X|\mathcal{G}_1)$$

4. Suppose X, Y are continuous random variables with joint density function $f(x, y)$.

   (a) Find $E(X|Y = y)$.

   (b) Recall that if a random variable Z is measurable with respect to $\sigma(Y)$ it may be written as $h(Y)$ for some function $h : \mathbb{R} \to \mathbb{R}$. Use this idea to write $E(X|Y)$ as a function of Y.

Written Problems (to be turned in): Due 11/28 at the start of class.

1. Assume that $X_1, X_2, ...$ are i.i.d. Bernoulli(p). This distribution does not satisfy the assumptions we made in the proof of the large deviation lower bound. Determine the large deviation rate function and show the lower bound explicitly using Stirling's approximation.

2. Assume that X and Y are independent wit the same distribution with $E|X| < \infty$. Show that $E(X|X + Y) = (X + Y)/2$.

3. Let X be a random variable with $E|X| < \infty$. In general the σ-field generated by $|X|$ will be smaller than the σ-field generated by X.

   (a) Suppose X is a discrete RV. Find an explicit formula for $E(X|\sigma(|X|))$.

   (b) Suppose the distribution of X is a.c. wrt Lebesgue measure with density f. Use your formula from part (a) to guess the form of $E(X|\sigma(|X|))$, then show that this formula is correct.

4. Define the **conditional variance** as

$$\text{Var }(X|\mathcal{F}) = E(X^2|\mathcal{F}) - E(X|\mathcal{F})^2$$

(a) Show that

$$\text{Var } X = E(\text{Var }(X|\mathcal{F})) + \text{Var }(E(X|\mathcal{F}))$$

(b) Let $Y_1, Y_2, ...$ be i.i.d. and N an independent integer positive integer valued random variable. Compute ES_N and Var S_N

5. Let $S_n = \sum_{k=1}^n X_k$ be a simple random walk, i.e. X_k are i.i.d. with $P(X_k = \pm 1) = 1/2$. Show that $S_n^2 - n$ is a martingale with respect to the natural filtration $\mathcal{F}_n = \sigma(X_1, ..., X_n), n \geq 1$.