PHY 354S: PROBLEM SET #2 **Due: Friday, February 10, 2017**

Do all problems. The starred question will be marked in detail. The other problems will be graded out of 2 to give you an incentive to do them: partial solutions will receive 1/2, correct or almost correct solutions will receive full marks.

1. Interestingly, it is possible to have two Lagrangians which give the same equations of motion but do not differ by a simple rescaling or addition of a time derivative. Consider a particle moving in one-dimension with position x and potential $V(x)$, governed by the Lagrangian,

$$L = \frac{1}{12}m^2\dot{x}^4 + m\dot{x}^2V(x) - V(x)^2.$$

Show that the resulting equation of motion is identical to that arising from the more traditional $L = \frac{1}{2}m\dot{x}^2 - V(x)$, despite the fact that the Lagrangians do not differ by a time derivative.

2. A mass m is attached to the end of a massless spring of spring constant k, as shown in the figure. The other end of the spring is fixed at the origin.

   (a) The spring/mass system is externally driven to rotate in the xy plane at constant angular velocity ω. Find the Lagrangian $L(r, \dot{r}, t)$, where r is the length of the spring. Find expressions for all conserved quantities.

   (b) Now suppose that the system is not externally driven but free to rotate. What are the generalized coordinates of the system now? Find the Lagrangian of the system and expressions for all conserved quantities.

   (c) Can the Hamiltonians in the two previous sections be written in terms of the generalized coordinates as $H = T + V$? Explain.

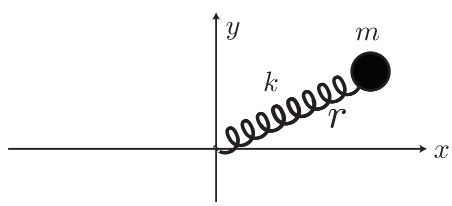

Figure 1: Problem 2

3. What components of momentum $\vec{P}$ and angular momentum $\vec{M}$ are conserved in motion in the following fields:

   (a) an infinite homogenous plane
   (b) an infinite homogeneous cylinder
   (c) an infinite homogenous prism
   (d) two point particles
   (e) an infinite homogenous half-plane
   (f) a homogenous cone
   (g) a homogeneous circular torus.

4. Consider a particle moving in the external potential field (it could be gravitational or electrostatic; it doesn't matter for this problem) of an infinite homogenous cylindrical helix (see figure). Find the conserved linear combination of components of $\vec{P}$ and $\vec{M}$.

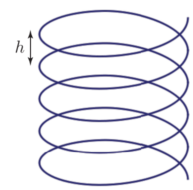

Figure 2: Problem 4

5. Consider a particle of mass m moving in a central force, $\vec{F}(\vec{r}) = f(r)\hat{r}$.

   (a) Show that
   $$\frac{d}{dt}\left(\vec{p}\times\vec{M}\right) = mf(r)\left(\dot{r}\vec{r} - r\dot{\vec{r}}\right)$$
   where $r \equiv |\vec{r}|$.

   (b) Show that
   $$\frac{d}{dt}\hat{r} = -\frac{1}{r^2}\left(\dot{r}\vec{r} - r\dot{\vec{r}}\right).$$

   (c) From the previous two results, show that in the case of a gravitational (or Coulomb) potential, $f(r) = -K/r^2$, there is an additional conserved quantity, the *Laplace-Runge-Lenz Vector*,
   $$\vec{A} = \vec{p}\times\vec{M} - mK\hat{r}$$
   which is not conserved for a general central potential. Show that $\vec{A}\cdot\vec{M} = 0$ and $|\vec{A}|^2$ is not independent of the other integrals of motion to argue that the three components of $\vec{A}$ only give one independent integral of the motion.

   In class, we will show how conservation of $\vec{A}$ allows us to solve the Kepler problem without doing any integrals.

6. * Consider the Atwood's machine shown in the figure, consisting of two massless pulleys supporting three masses m_1, m_2 and m_3 via massless strings. Write the Lagrangian for the system in terms of the generalized coordinates x_1 and x_2. Identify a symmetry of the corresponding Lagrangian which is not simple time or space translation and use Noether's theorem to find the corresponding conserved quantity. Verify directly from the equations of motion that this quantity is conserved.

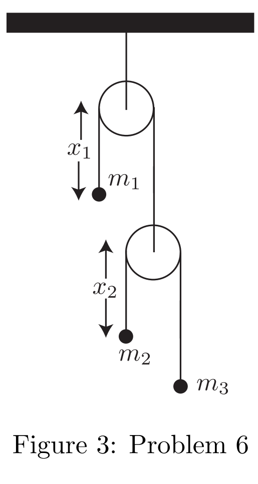

Figure 3: Problem 6