PHY 354S: PROBLEM SET #5 **Due: Wednesday, April 5, 2017**

Do all problems. The two starred questions will be marked in detail. The other problems will be graded out of 2 to give you an incentive to do them: partial solutions will receive 1/2, correct or almost correct solutions will receive full marks.

1. *The fast top:* Consider a heavy symmetry top of mass M, pinned at point P which is a distance ℓ from the centre of mass, as discussed in class. The principal moments of inertia about P are I_1, I_1 and I_3. The top is spun with initial conditions $\dot{\phi} = 0$ and $\theta = \theta_0$. Show that θ obeys the equation of motion

$$I_1 \ddot{\theta} = -\frac{dV_{\text{eff}}(\theta)}{d\theta}$$

where

$$V_{\text{eff}}(\theta) = \frac{I_3^2 \Omega_3^2}{2I_1} \frac{(\cos\theta - \cos\theta_0)^2}{\sin^2\theta} + Mg\ell\cos\theta.$$

Suppose the top is spinning very fast, so that $I_3 \omega_3 \gg \sqrt{Mg\ell I_1}$. Show that θ_0 is close to the minimum of $V_{\text{eff}}(\theta)$, and use this fact to deduce that the top nutates with frequency $\omega \simeq \Omega_3 I_3 / I_1$ and draw the subsequent motion.

2. Landau & Lifschitz, pg. 113, Problem 2

3. A system with two degrees of freedom x and y has the Lagrangian

$$L = x\dot{y} + y\dot{x}^2 + \dot{x}\dot{y}.$$

Derive the Euler-Lagrange equations. Find the Hamiltonian $H(x, y, p_x, p_y)$. Derive Hamilton's equations and show that they are equivalent to Lagrange's equations.

4. The Lagrangian for the heavy symmetric top is

$$L = \frac{1}{2} I_1 \left(\dot{\theta}^2 + \dot{\phi}^2 \sin^2\theta \right) + \frac{1}{2} I_3 \left(\dot{\psi} + \dot{\phi}\cos\theta \right)^2 - Mg\ell\cos\theta.$$

Obtain the momenta p_θ, p_ϕ and p_ψ, and the Hamiltonian $H(\theta, \phi, \psi, p_\theta, p_\phi, p_\psi)$. Derive Hamilton's equations.

5. * A rigid lamina (i.e. a 2 dimensional object) has principal moments of inertia about the centre of mass of

$$I_1 = (\mu^2 - 1), \; I_2 = (\mu^2 + 1), \; I_3 = 2\mu^2.$$

   (a) Show, using Euler's equations, that in the body-fixed frame, the component of the angular velocity in the plane of the lamina (i.e. $\sqrt{\Omega_1^2 + \Omega_2^2}$) is constant in time.

   (b) Choose the initial angular velocity to be $\vec{\Omega} = \mu N \hat{x}_1 + N \hat{x}_3$. Define $\tan\alpha = \Omega_2/\Omega_1$, which is the angle the component of Ω in the plane of the lamina makes with $\hat{x}_1$. Show that it satisfies

$$\ddot{\alpha} = \dot{\Omega}_3$$

   and from this show that

$$\ddot{\alpha} = -\frac{1}{\mu^2} \left(\Omega_1^2 + \Omega_2^2 \right) \sin\alpha \cos\alpha = -N^2 \cos\alpha \sin\alpha.$$

   Show that the solution to the motion is

$$\vec{\Omega}(t) = \mu N (\cosh Nt)^{-1} \hat{x}_1 + \mu N \tanh Nt \hat{x}_2 + N (\cosh Nt)^{-1} \hat{x}_3.$$

   (NB It is enough to check that this is the solution; you do not need to solve the differential equation).

6. * A particle with mass m, position $\vec{r}$ and momentum $\vec{p}$ has angular momentum $\vec{M} = \vec{r} \times \vec{p}$.

(a) Evaluate $\{x_j, M_k\}$, $\{p_j, M_k\}$, $\{M_j, M_k\}$ and $\{M_i, \vec{M}^2\}$.

(b) Consider a particle moving of mass m moving in a central potential $V(r) = -K/r$. Recall the Runge-Lenz vector

$$\vec{A} = \vec{p} \times \vec{M} - mK\hat{r}.$$

Show that

$$\{M_i, A_j\} = -\epsilon_{ijk} A_k,\ \{A_i, A_j\} = 2Hm\epsilon_{ijk} M_k.$$

Prove using Poisson brackets that $\vec{A}$ is conserved. (Also note that $\vec{A}$, $\vec{M}$ and H form a closed algebra under the Poisson bracket.)