Isomorphisms and homomorphisms

1. For each of the following groups, find a familiar group that it is isomorphic to and write down an isomorphism.

   (a) $D_3/\{I, D, D^2\}$

   (b) $(\mathbb{R}\backslash\{0\}, \cdot)/\{-1, 1\}$

   (c) $\mathrm{GL}_2(\mathbb{R})/\mathrm{SL}_2(\mathbb{R})$ where $\mathrm{SL}_2(\mathbb{R})$ denotes the subgroup consisting of 2×2 real matrices of determinant 1.

   (d) $\left\{ \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} : x \in \mathbb{R} \right\}$ under multiplication (First check that this is in fact a group.)

2. Find a non-trivial homomorphism from $(\mathbb{R}, +)/\mathbb{Z}$ to $\mathrm{GL}_2(\mathbb{R})$. Describe the image and the kernel. (Being non-trivial means that not everything is sent to the neutral element.)

3. Let n be an integer ≥ 2. Find all homomorphisms from $(\mathbb{Z}/n\mathbb{Z}, +)$ to $(\mathbb{R}, +)/\mathbb{Z}$.

4. Find all homomorphisms from $(\mathbb{Z}, +)$ to $(\mathbb{R}, +)/\mathbb{Z}$.

5. Find a subgroup of $(\mathbb{R}, +)/\mathbb{Z}$ that is isomorphic to $(\mathbb{Z}, +)$?

6. Check that $\mathbb{Z} \times \mathbb{Z}$ is a group under addition defined by $(a, b) + (c, d) = (a + c, b + d)$. Find a subgroup of $(\mathbb{R}, +)/\mathbb{Z}$ that is isomorphic to $(\mathbb{Z} \times \mathbb{Z}, +)$.

7. Find all homomorphisms from $(\mathbb{Z}/6\mathbb{Z}, +)$ to $(\mathbb{Z}/9\mathbb{Z}, +)$.
   Generalize this: describe all homomorphisms from $(\mathbb{Z}/m\mathbb{Z}, +)$ to $(\mathbb{Z}/n\mathbb{Z}, +)$.

8. Let G and K be groups with more than one element each.
   Prove or disprove: If there is a non-trivial homomorphism from G to K, then there is a non-trivial homomorphism from K to G.