Math 4107

Some more practice problems for Exam 2 (in addition to the old exam posted on Piazza).

1. Let G be a non-abelian group. Show that G must contain a non-trivial abelian subgroup. Must it contain a non-trivial normal subgroup?

2. Let G be a group acting on the set $\{1, 2, \ldots, n\}$. Show that the map that sends $g \in G$ to the function $i \mapsto g \cdot i$ for $i \in \{1, 2, \ldots, n\}$ is a group homomorphism from G to S_n.

3. Prove that every finite group is isomorphic to a subgroup of a symmetric group. This is called Cayley's theorem.
   Prove that every finite group is isomorphic to a subgroup of an alternating group.

4. Let G and K be finite groups. Prove that if $|G|$ and $|K|$ are relatively prime then any homomorphism $\phi : G \to K$ must be the trivial homomorphism, i.e. for any $g \in G$ the image $\phi(g)$ must the neutral element in K.

5. Let $\sigma = (1\,2\,3\,4)(3\,4\,5\,6) \in S_6$.

   (a) Is σ even or odd? ______

   (b) Write σ as a product of disjoint cycles: ____________

   (c) What is the order of σ? ______

   (d) What is the number of inversions of σ? ______

   (e) Find an element $\tau \in S_6$ such that $\tau^{-1}\sigma\tau = (1\,3\,5)(2\,4\,6)$. ____________

   (f) How many elements of S_6 commute with σ? ____________

   (g) What is the size of the conjugacy class of σ? ____________

6. How many ways are there to color the edges of a square with k-colors, where two colorings are considered the same if they map to each other using rotations and reflections.

7. Let G be a group of order 35 acting on a set S of size 16. Show that there must be a fixed point (fixed by all elements $g \in G$).

8. Let P be a triangular prism $\Delta \times I \subset \mathbb{R}^3$, where Δ is an equilateral triangle and I is a line segment perpendicular to Δ. Find the order of the the group of symmetries (rotations and reflections) of P.