Math 4107

Some more practice problems for Exam 3 (in addition to the old exam posted on Piazza).

1. Prove that groups of order 35 are cyclic.
2. Give an example of a non-commutative ring R and two elements $a, b \in R$ such that $ab = 0$ but $ba \neq 0$.
3. Find all maximal ideals in $\mathbb{Z}$. Justify your answer.
4. Let p be a prime integer. Prove that if $p \equiv 3 \pmod 4$, then p is a prime element in $\mathbb{Z}[i]$.
5. Construct a field with 27 elements.
6. Let R be a ring and let I be an ideal in R. The *annihilator* of I is defined as $\mathrm{Ann}(I) = \{x \in R : xa = 0 \text{ for all } a \in I\}$.
   (a) Show that $\mathrm{Ann}(I)$ is an ideal.
   (b) Let $R = \mathbb{Z}/12\mathbb{Z}$. Find $\mathrm{Ann}(I)$ where I is the ideal generated by $[3]$ in R.
7. Compute a greatest common divisor of $f = x^2+2$ and $g = x^4+2x^3+x^2+2$ in $\mathbb{Z}/3\mathbb{Z}[x]$.
8. Let R be a non-commutative ring, and let $I \subset R$ be an ideal such that $ab - ba \in I$ for any $a, b \in R$. Show that R/I is a commutative ring.
9. In $\mathbb{Z}[i]$, factor 13 into prime elements.
10. Prove that for a polynomial f of degree n over a field F, there are at most n solutions to $f(x) = 0$ in $\mathbb{F}$.

Challenge problems:

1. Let $\mathbb{F}$ be a field. Prove that every finite subgroup of $\mathbb{F}^*$ is cyclic. (Hint: consider the roots of polynomials of the form $x^k - 1$.)
2. Show that there is no commutative ring R such that $(R, +) \cong (\mathbb{Q}/\mathbb{Z}, +)$.