Natural Numbers

Definitions: We define $0 = \varnothing$ and the *successor* of x as $x^+ = x \cup \{x\}$.
(Think of x^+ as $x + 1$.)

Successor set (or inductive set): a set A such that $0 \in S$ and $x^+ \in A$ for every $x \in A$.

Axiom of Infinity: There exists a successor set.

Exercises

1. Prove that the intersection of a non-empty collection of successor sets is a successor set itself.
2. Prove that there exists a unique successor set that is a subset of every successor set.

Definition of natural numbers
The unique minimal successor set is called the *set of natural numbers* and is denoted $\mathbb{N}$ or ω. A *natural number* is defined to be an element of ω.

Properties of ω (The Peano Axioms)

(I) $0 \in \mathbb{N}$

(II) $\forall n \in \mathbb{N}, n^+ \in \mathbb{N}$

(III) $\forall S \subset \omega, [0 \in S \wedge (\forall n \in S, n^+ \in S)] \Rightarrow S = \omega$.

(IV) $\forall n \in \omega, n^+ \neq 0$.

(V) $\forall n, m \in \omega, (n^+ = m^+ \Rightarrow n = m)$.

Properties (I)-(IV) follow easily from the definition of ω, and (V) can also be proven from the definition. (See Halmos.) Property (III) is called the *Principle of Mathematical Induction.*

Addition Define $x{+}1 = x^+$, $x{+}2 = (x{+}1)^+$, $x{+}3 = (x{+}2)^+, \ldots, x{+}(n{+}1) = (x{+}n)^+$, and so on inductively. (See the "Recursion Theorem" in Halmos for a careful treatment.)

Multiplication Define $2x = x + x$, $3x = 2x + x, \ldots, (n + 1)x = nx + x$, and so on.

Ordering Define $i < j$ if $i \in j$.

(see back page)

Exercises

3. For natural numbers m and n, what does $m \cap n$ and $m \cup n$ mean in terms of the "usual" operations on integers?

4. For which natural numbers m and n is the set difference $m - n$ also a natural number?

5. Describe a successor set that is not ω.