SYS 6003: Optimization	**Fall 2016**

Lecture 1

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Given a vector space $\mathcal{X}$, a norm is a function $\|\cdot\| : \mathcal{X} \to \mathbb{R}$ that satisfies the following properties:

1. For any $\mathbf{x} \in \mathcal{X}$ and scalar α, $\|\alpha\mathbf{x}\| = |\alpha| \cdot \|\mathbf{x}\|$
2. For any $\mathbf{x}, \mathbf{y} \in \mathcal{X}$, $\|\mathbf{x} + \mathbf{y}\| \leq \|\mathbf{x}\| + \|\mathbf{y}\|$
3. If $\|\mathbf{x}\| = 0$ then $\mathbf{x} = \mathbf{0}$

Note that property 1 implies that $\|\mathbf{0}\| = 0$, and properties 1 and 2 imply that for any $\mathbf{x} \in \mathcal{X}$, $\|\mathbf{x}\| \geq 0$.

We now give definitions for some specific vector norms.

Definition 1 *For any* $\mathbf{x} = (x_1, \ldots, x_d)^\top \in \mathbb{R}^d$,

$$\begin{aligned} \|\mathbf{x}\|_2 &= \sqrt{\textstyle\sum_{i=1}^d x_i^2} \quad (\ell_2 \text{ norm}) \\ \|\mathbf{x}\|_1 &= \textstyle\sum_{i=1}^d |x_i| \quad (\ell_1 \text{ norm}) \\ \|\mathbf{x}\|_\infty &= \max_{1 \leq i \leq d} |x_i| \quad (\ell_\infty \text{ norm}) \end{aligned}$$

Two norms $\|\cdot\|$ and $\|\cdot\|^*$ are said to be *equivalent* if there exist positive real numbers c and C such that for any $\mathbf{x} \in \mathcal{X}$, $c\|\mathbf{x}\|^* \leq \|\mathbf{x}\| \leq C\|\mathbf{x}\|^*$. In fact, if the vector space has finite dimensions (e.g. $\mathbb{R}^d$), then all norms on this vector space are equivalent. In particular, we have the following inequalities:

$$\begin{aligned} \|\mathbf{x}\|_2 &\leq \|\mathbf{x}\|_1 \leq \sqrt{d}\,\|\mathbf{x}\|_2 \\ \|\mathbf{x}\|_\infty &\leq \|\mathbf{x}\|_2 \leq \sqrt{d}\,\|\mathbf{x}\|_\infty \\ \|\mathbf{x}\|_\infty &\leq \|\mathbf{x}\|_1 \leq d\,\|\mathbf{x}\|_\infty. \end{aligned}$$

Next we define the Ball $B(\mathbf{x}, \delta)$ centered at a given point $\mathbf{x} \in \mathcal{X}$ with radius $\delta > 0$, which can be viewed as the neighborhood of $\mathbf{x}$:

$$B(\mathbf{x}, \delta) = \{\mathbf{y} \in \mathcal{X} : \|\mathbf{y} - \mathbf{x}\|_2 \leq \delta\}.$$

In this course, we consider the set in the Euclidean space, i.e., $\mathbb{R}$ or $\mathbb{R}^d$.

Definition 2 (Interior point) $\mathbf{x} \in \mathcal{C}$ *is an interior point of set* $\mathcal{C}$ *if there exist a* $\delta > 0$*, such that* $B(\mathbf{x}, \delta) \subseteq \mathcal{C}$.

Definition 3 (Interior of a set) *The set of all points interior to* $\mathcal{C}$ *is called to the interior of* $\mathcal{C}$*, and is denoted by* **int** $\mathcal{C}$.

Definition 4 (Open set) *A set $\mathcal{C}$ is open if* **int** $\mathcal{C} = \mathcal{C}$, *i.e., every point in set $\mathcal{C}$ is an interior point.*

Definition 5 (Closed set) *A set $\mathcal{C}$ is closed if its complement $\mathbb{R}^d \setminus \mathcal{C}$ is open.*

Definition 6 (Boundary point) $\mathbf{x}$ *is a boundary point of set $\mathcal{C}$ if for any $\delta > 0$, there exist $\mathbf{y} \in \mathcal{C}$, and $\mathbf{z} \notin \mathcal{C}$ such that $\mathbf{y} \in B(\mathbf{x}, \delta)$, $\mathbf{z} \in B(\mathbf{x}, \delta)$.*