SYS 6003: Optimization I	Fall 2016

Homework 1

Handed Out: September 7^{th}, 2016 — *Due: September* 19^{th}, 2016

- Feel free to talk to other students in the class when doing the homework. You should, however, write down your solution yourself. Please try to keep the solution brief and clear.
- You will write your solution in Latex and submit the printed pdf file in class.
- The first page of your submission should include the pledge followed by your signature.
- The homework is due at 3:30 PM (before the class) on the due date.

1. (10 points) Find the minimizer and maximizer of the function

$$f(x_1, x_2) = x_1 + x_2$$

over the set

$$\mathcal{D} = \{\mathbf{x} \in \mathbb{R}^2 : x_1^2 + x_2^2 \leq 1, x_1 \geq 0, x_2 \geq 0\}.$$

2. (10 points) Does there exist a minimizer and maximizer for the function

$$f(x) = x_1^3 + 2x_2^2 + 3x_3 + 4x_4^5$$

over the set

$$\mathcal{D} = \{\mathbf{x} \in \mathbb{R}^4 : x_1^2 + 2x_2^2 + 3x_3^2 + 4x_4^2 \leq 1\} \quad ?$$

Why?

3. (10 points) Consider the unconstrained minimization problem:

$$\min_{\mathbf{x} \in \mathbb{R}^2} f(\mathbf{x}) = x_1^2 - x_1 x_2 + x_2^2 - 3x_2. \tag{1}$$

Find a local minimum.

4. (10 points) Show that a set is convex if and only if its intersection with any line is convex.

5. (10 points) Show that the convex hull of a set $\mathcal{S}$ is the intersection of all convex sets that contain $\mathcal{S}$.

6. (10 points) Let $\mathcal{C}$ be the solution set of a quadratic inequality:

$$\mathcal{C} = \{\mathbf{x} \in \mathbb{R}^d : \mathbf{x}^\top \mathbf{A}\mathbf{x} + \mathbf{b}^\top \mathbf{x} + c \leq 0\},$$

where $\mathbf{b} \in \mathbb{R}^d$,$c \in \mathbb{R}$, and $\mathbf{A} \in \mathbb{R}^{d \times d}$ is a symmetric matrix . Show that $\mathcal{C}$ is convex if $\mathbf{A} \succ 0$.

7. (10 points) If f is an affine function and $\mathcal{S}$ is convex, show that the image of $\mathcal{S}$ under f, i.e., $f(\mathcal{S})$ is convex.

8. (10 points) Show that the set of points closer to a given point $\mathbf{x}_0 \in \mathbb{R}^d$ than to a given set $\mathcal{S} \in \mathbb{R}^d$, i.e.,

$$\{\mathbf{x} \in \mathbb{R}^d : \|\mathbf{x} - \mathbf{x}_0\|_2 \leq \|\mathbf{x} - \mathbf{y}\|_2 \text{ for all } \mathbf{y} \in \mathcal{S}\}.$$

   is convex.

9. (10 points) Consider the set $\{\mathbf{x} \in \mathbb{R}^d : \mathbf{x} + \mathcal{S}_2 \subseteq \mathcal{S}_1\}$, where $\mathcal{S}_1, \mathcal{S}_2 \in \mathbb{R}^d$ with $\mathcal{S}_1$ being convex. Show that this is a convex set.

10. (10 points) Show that if $\mathcal{S}_1$ and $\mathcal{S}_2$ are convex sets in $\mathbb{R}^m \times \mathbb{R}^n$, then so is their partial sum:

$$\mathcal{S} = \big\{(\mathbf{x}, \mathbf{y}_1 + \mathbf{y}_2) : \mathbf{x} \in \mathbb{R}^m, \mathbf{y}_1, \mathbf{y}_2 \in \mathbb{R}^n, (\mathbf{x}, \mathbf{y}_1) \in \mathcal{S}_1, (\mathbf{x}, \mathbf{y}_2) \in \mathcal{S}_2\big\}.$$