SYS 6003: Optimization **Fall 2016**

Lecture 9

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We continue to study convexity preserving operations for convex functions.

1. Pointwise Maximum: If $f_1, f_2, ..., f_m$ are convex, then $f(\mathbf{x}) = \max\{f_1(\mathbf{x}), ..., f_m(\mathbf{x})\}$ is convex on $\mathbf{dom}\ f = \cap_{i=1}^m \mathbf{dom}\ f_i$
Proof: For any $\mathbf{x}, \mathbf{y} \in \mathbf{dom}\ f$, and any $\alpha \in [0, 1]$, we have

$$\begin{aligned} f(\alpha\mathbf{x} + (1-\alpha)\mathbf{y}) &= \max\{f_1(\alpha\mathbf{x} + (1-\alpha)\mathbf{y}), ..., f_m(\alpha\mathbf{x} + (1-\alpha)\mathbf{y})\} \\ &\le \max\{\alpha f_1(\mathbf{x}) + (1-\alpha)f_1(\mathbf{y}), ..., \alpha f_m(\mathbf{x}) + (1-\alpha)f_m(\mathbf{y})\} \\ &= \max_{1\le i\le m}[\alpha f_i(\mathbf{x}) + (1-\alpha)f_i(\mathbf{y})]. \end{aligned}$$

Note that

$$\begin{aligned} \max_{1\le i\le m}[\alpha f_i(\mathbf{x}) + (1-\alpha)f_i(\mathbf{y})] &\le \alpha \max_{1\le i\le m}\{f_i(\mathbf{x})\} + (1-\alpha) \max_{1\le i\le m}\{f_i(\mathbf{y})\} \\ &= \alpha f(\mathbf{x}) + (1-\alpha)f(\mathbf{y}). \end{aligned}$$

Thus, by definition, $f(\mathbf{x})$ is convex. ■

Example 1 (Piecewise affine function)

$$f(\mathbf{x}) = \max_{1\le i\le m} \mathbf{a}_i^\top \mathbf{x} + b_i,$$

where $\mathbf{a}_i, \mathbf{x} \in \mathbb{R}^d$ *and* $b_i \in \mathbb{R}$ *is convex.*

Example 2 (Sum of r largest components of $\mathbf{x} \in \mathbb{R}^d, 1 \le r \le d$) *Let* $g_{\pi^k}(\mathbf{x}) = \sum_{i=1}^r \mathbf{x}_{\pi^k(i)}$*, where* π^k *is a permutation of* $\{1, ..., d\}$*. Note that there are at most* $d!$ *permutations. The sum of* r *largest components of* $\mathbf{x}$*:*

$$f(\mathbf{x}) = \max_{1\le k\le d!} g_{\pi^k}(\mathbf{x})$$

is a convex function.

2. Pointwise Supremum: If $f(\mathbf{x}, \mathbf{y})$ is convex in $\mathbf{x}$ for each $\mathbf{y} \in \mathcal{A}$, then

$$g(\mathbf{x}) = \sup_{\mathbf{y}\in\mathcal{A}} f(\mathbf{x}, \mathbf{y})$$

is convex.
Proof: For any $\mathbf{x}_1, \mathbf{x}_2 \in \mathbb{R}^d$, and any $\alpha \in [0, 1]$, we have

$$g(\alpha\mathbf{x}_1 + (1-\alpha)\mathbf{x}_2) = \sup_{\mathbf{y}\in\mathcal{A}} f(\alpha\mathbf{x}_1 + (1-\alpha)\mathbf{x}_2, \mathbf{y}).$$

For any given $\epsilon > 0$, there exist a $\mathbf{y}_\epsilon$, such that

$$\sup_{\mathbf{y}\in\mathcal{A}} f(\alpha\mathbf{x}_1 + (1-\alpha)\mathbf{x}_2, \mathbf{y}) \le f(\alpha\mathbf{x}_1 + (1-\alpha)\mathbf{x}_2, \mathbf{y}_\epsilon) + \epsilon.$$

By convexity of f in $\mathbf{x}$, we have

$$\begin{aligned} f(\alpha\mathbf{x}_1 + (1-\alpha)\mathbf{x}_2, \mathbf{y}_\epsilon) + \epsilon &\le \alpha f(\mathbf{x}_1, \mathbf{y}_\epsilon) + (1-\alpha) f(\mathbf{x}_2, \mathbf{y}_\epsilon) + \epsilon \\ &\le \alpha \sup_{\mathbf{y}\in\mathcal{A}} f(\mathbf{x}_1, \mathbf{y}) + (1-\alpha) \sup_{\mathbf{y}\in\mathcal{A}} f(\mathbf{x}_2, \mathbf{y}) + \epsilon \\ &= \alpha g(\mathbf{x}_1) + (1-\alpha) g(\mathbf{x}_2) + \epsilon. \end{aligned}$$

Let $\epsilon \to 0$, then we obtain

$$g(\alpha\mathbf{x}_1 + (1-\alpha)\mathbf{x}_2) \le \alpha g(\mathbf{x}_1) + (1-\alpha) g(\mathbf{x}_2).$$

By definition, $g(\mathbf{x})$ is convex. ■

Example 3 $f(\mathbf{x}) = \sup_{\mathbf{y}\in\mathcal{C}} \|\mathbf{x} - \mathbf{y}\|_2$ *is convex. To see that, let*

$$\begin{aligned} g(\mathbf{x}, \mathbf{y}) &= \|\mathbf{x} - \mathbf{y}\|_2 && (1) \\ &= \left\| \mathbf{A} \begin{bmatrix} \mathbf{x} \\ \mathbf{y} \end{bmatrix} \right\|_2, && (2) \end{aligned}$$

where $\mathbf{A}$ *is defined as*

$$\mathbf{A} = \left[\mathbf{I}, -\mathbf{I}\right]. \quad (3)$$

Since $g(\mathbf{x}, \mathbf{y})$ *is a composition of* ℓ_2 *norm with affine function, it is convex in* $(\mathbf{x}, \mathbf{y})$*. Then for each* $\mathbf{y} \in \mathcal{C}$*,* $g(\mathbf{x}, \mathbf{y})$ *is convex in* $\mathbf{x}$*. Thus, from the property of pointwise supremum,*

$$f(\mathbf{x}) = \sup_{\mathbf{y}\in\mathcal{C}} g(\mathbf{x}, \mathbf{y})$$

is convex.

Example 4 (Maximum eigenvalue of a symmetric matrix) *For any* $\mathbf{A} \in \mathbb{R}^{d\times d}$*, its maximum eigenvalue*

$$\lambda(\mathbf{A}) = \sup_{\|\mathbf{x}\|_2 = 1, \mathbf{x}\in\mathbb{R}^d} \mathbf{x}^\top \mathbf{A}\mathbf{x},$$

is convex in $\mathbf{A}$*.*

3. Minimization over Some Variables: If $f(\mathbf{x}, \mathbf{y})$ is convex in $(\mathbf{x}, \mathbf{y})$ and $\mathcal{C}$ is a convex set, then

$$g(\mathbf{x}) = \inf_{\mathbf{y}\in\mathcal{C}} f(\mathbf{x}, \mathbf{y})$$

is convex.

Example 5 *The function* $\text{dist}(\mathbf{x}, \mathcal{S}) = \inf_{\mathbf{y} \in \mathcal{S}} \ ||\mathbf{x} - \mathbf{y}||_2$*, where* $\mathcal{S}$ *is a convex set, is convex.*

4. Perspective of a function: If $f : \mathbb{R}^d \to \mathbb{R}$ is convex, then its perspective function $g : \mathbb{R}^d \times \mathbb{R} \to \mathbb{R}$ defined by $g(\mathbf{x}, t) = tf(\mathbf{x}/t)$, with domain

$$\mathbf{dom}\ g = \{(\mathbf{x}, t) : t > 0,\ \mathbf{x}/t \in \mathbf{dom}\ f\}$$

is convex.

Proof: Consider the epigraphs of f and g:

$$\begin{aligned}
\mathbf{epi}(f) &= \{(\mathbf{x}, s) : f(\mathbf{x}) \le s, \mathbf{x} \in \mathbf{dom}\ f\}, \\
\mathbf{epi}(g) &= \{(\mathbf{x}, t, s) : f(\mathbf{x}/t) \le s/t, (\mathbf{x}, t) \in \mathbf{dom}\ g\} \\
&= \{(\mathbf{x}, t, s) : f(\mathbf{x}/t) \le s/t, \mathbf{x}/t \in \mathbf{dom}\ f, t > 0\}.
\end{aligned}$$

Thus, $(\mathbf{x}, t, s) \in \mathbf{epi}(g)$ if and only if $(\mathbf{x}/t, s/t) \in \mathbf{epi}(f)$. Define perspective function:

$$\text{perp}((\mathbf{x}, t, s)) = (\mathbf{x}, s)/t.$$

As we can see, $\mathbf{epi}(f) = \text{perp}(\mathbf{epi}(g))$. Since f is convex, its epigraph, $\mathbf{epi}(f)$, is also convex. Thus, $\mathbf{epi}(g)$, which is the inverse image of $\mathbf{epi}(f)$ under the perspective function, is convex. We then know that g is a convex function. ■