IERG6120 Coding for Distributed Storage Systems Lecture 17 - 01/11/2016

An Exact-Repair $(4, 2, 3)$ MSR Code

Lecturer: Kenneth Shum *Scribe: Qiaoqiao Zhou*

In this lecture, we look at a special help-by-transfer MSR code in details. More specifically, we will go through an exact-repair (4,2,3) regenerating code.

First, recall the definition of minimum storage regenerating (MSR) point, which is stated as follow:

Lemma 1 (MSR point)**.** *The minimum storage regenerating (MSR) point is*

$$\alpha_{\text{MSR}} = \frac{B}{k}, \quad d\beta_{\text{MSR}} = \frac{dB}{k(d+1-k)} \tag{1}$$

Consider an exact-repair $(4, 2, 3)$ regenerating code, where $n = 4, k = 2, d = 3, B = k(d+1-k) = 4, \alpha = 3, \beta = 1$. Therefore, by Lemma 1, this is a MSR code. Suppose the information symbols are $\mathsf{A}, \mathsf{B}, \mathsf{C}, \mathsf{D}$. The explicit code structure is

$$\begin{array}{lll} \text{Node 1:} & \mathsf{A} & \mathsf{B}+\mathsf{C} \\ \text{Node 2:} & \mathsf{B} & \mathsf{C}+\mathsf{D} \\ \text{Node 3:} & \mathsf{C} & \mathsf{D}+\mathsf{A} \\ \text{Node 4:} & \mathsf{D} & \mathsf{A}+\mathsf{B} \end{array}$$

We first check the $(4, 2)$ recover property. Due to cyclic symmetry, we only need to consider the following 2 cases:

- 2 consecutive nodes: For example, if we choose node 1 and 2, we can download $\mathsf{A}, \mathsf{B}, \mathsf{B}+\mathsf{C}, \mathsf{C}+\mathsf{D}$ from them. A and B can be recover trivially. C and D can also be recovered by

$$\begin{aligned} \mathsf{C} &= \mathsf{B}+\mathsf{B}+\mathsf{C} \\ \mathsf{D} &= \mathsf{C}+\mathsf{C}+\mathsf{D} \end{aligned} \tag{2}$$

- 2 non-consecutive nodes: For example, if we choose node 1 and 3, we can download $\mathsf{A}, \mathsf{B}+\mathsf{C}, \mathsf{C}, \mathsf{D}+\mathsf{A}$ from them. A and C can be recover trivially. B and D can also be recovered by

$$\begin{aligned} \mathsf{B} &= \mathsf{B}+\mathsf{C}+\mathsf{C} \\ \mathsf{D} &= \mathsf{A}+\mathsf{D}+\mathsf{A} \end{aligned} \tag{3}$$

Thus, this is a MDS code and maintains the recover property.

Then, we check the exact-repair property. Suppose node 1 fails, we can collect $\mathsf{B}, \mathsf{C}, \mathsf{A}+\mathsf{B}$ from node 2, 3, 4, respectively. Note that the symbol sent to the new node can be read directly from the memory, without doing any arithmetic operation, i.e., no network coding on the transmitting side. $\mathsf{B}+\mathsf{C}$ can be recovered trivially after we get B and C. A can be recovered by $\mathsf{A} = \mathsf{B}+\mathsf{A}+\mathsf{B}$. Therefore, node 1 can be recovered exactly. In the following, we illustrate how interference alignment can be applied to show the repairability.

First, we change the basis as follows: Assume the 4 information symbols are $\mathsf{P}, \mathsf{Q}, \mathsf{R}, \mathsf{S}$. Node 1 store P, Q and node 2 store R, S. Then, we have the following relationship

$$\begin{aligned} \mathsf{A} &= \mathsf{P} \\ \mathsf{B}+\mathsf{C} &= \mathsf{Q} \\ \mathsf{B} &= \mathsf{R} \\ \mathsf{C}+\mathsf{D} &= \mathsf{S} \end{aligned} \tag{4}$$

After changing the basis, we have

$$\begin{array}{lcc} \text{Node 1:} & \mathsf{P} & \mathsf{Q} \\ \text{Node 2:} & \mathsf{R} & \mathsf{S} \\ \text{Node 3:} & \mathsf{R}+\mathsf{Q} & \mathsf{P}+\mathsf{Q}+\mathsf{R}+\mathsf{S} \\ \text{Node 4:} & \mathsf{R}+\mathsf{Q}+\mathsf{S} & \mathsf{P}+\mathsf{R} \end{array}$$

This becomes a systematic code.

Suppose node 1 fails. According to the previous discussion, we are able to repair node 1 by collecting $\mathsf{B}, \mathsf{C}, \mathsf{A}+\mathsf{B}$ from node 2, 3, 4, respectively. Similarly, we want to repair node 1 by collecting $\mathsf{R}, \mathsf{R}+\mathsf{Q}, \mathsf{P}+\mathsf{R}$ from node 2, 3, 4, respectively. Observing that the symbols R is an *interference* to node 1, since he only want to recover P, Q. On the other hand, each of the other two received symbols can be written as a linear combination of the interference symbol and the desired symbols. Therefore, we just regarded R as an interference and subtract it from the remaining 2 symbols. More specifically,

$$\begin{aligned} \mathsf{R} &\Longrightarrow \text{interference} \\ \mathsf{R}+\mathsf{Q} &\Longrightarrow \mathsf{Q}+\mathsf{R} \\ \mathsf{P}+\mathsf{R} &\Longrightarrow \mathsf{P}+\mathsf{R} \end{aligned} \tag{5}$$

Suppose node 2 fails. we can download $\mathsf{Q}, \mathsf{R}+\mathsf{Q}, \mathsf{R}+\mathsf{Q}+\mathsf{S}$ from user 1, 3, 4, respectively. The repair scheme is as follows:

$$\begin{aligned} \mathsf{Q} &\Longrightarrow \text{interference} \\ \mathsf{R}+\mathsf{Q} &\Longrightarrow \mathsf{R}+\mathsf{Q} \\ \mathsf{R}+\mathsf{Q}+\mathsf{S} &\Longrightarrow \mathsf{R}+\mathsf{S}+\mathsf{Q} \end{aligned} \tag{6}$$

Suppose node 3 fails. we can download $\mathsf{P}, \mathsf{S}, \mathsf{R}+\mathsf{Q}+\mathsf{S}$ from user 1, 2, 4, respectively. The repair scheme is as follows:

$$\begin{aligned} \mathsf{P} &\Longrightarrow \mathsf{P}+\mathsf{Q}+\mathsf{R}+\mathsf{S}+\mathsf{R}+\mathsf{Q}+\mathsf{S} \\ \mathsf{S} &\Longrightarrow \mathsf{R}+\mathsf{Q}+\mathsf{R}+\mathsf{Q}+\mathsf{S} \\ \mathsf{R}+\mathsf{Q}+\mathsf{S} &\Longrightarrow \text{interference} \end{aligned} \tag{7}$$

Suppose node 4 fails. we can download $\mathsf{P}, \mathsf{R}, \mathsf{P}+\mathsf{Q}+\mathsf{R}+\mathsf{S}$ from user 1, 2, 3, respectively. The repair scheme is as follows:

$$\begin{aligned} \mathsf{P} &\Longrightarrow \text{interference} \\ \mathsf{R} &\Longrightarrow \mathsf{P}+\mathsf{R}+\mathsf{P} \\ \mathsf{P}+\mathsf{Q}+\mathsf{R}+\mathsf{S} &\Longrightarrow \mathsf{R}+\mathsf{Q}+\mathsf{S}+\mathsf{P} \end{aligned} \tag{8}$$

Therefore, the code satisfies the repairability.

In this small example, we can frame the repair procedure as the following story. The failure node wants to recover two particular symbols X and Y. From the three received symbols, one of them can be regarded as an *interference*. Each of the other two received symbols can be written as a linear combination of the interference symbol and X and Y. We can solve for symbols X and Y after subtracting off the interference term. This is a paradigm in constructing exact-repair MSR regenerating code. The encoding is conjured in such a way that the *interference* can be confined to a subspace of small dimension.

References

[1] C. Suh and K. Ramchandran, "Exact-repair MDS code construction using interference alignment," *IEEE Trans. Inf. Theory*, vol. 57, no. 3, pp.1425-1442, Mar. 2011.