Chapter 4

Taylor and Laurent Series

4.1. Taylor Series

4.1.1. Taylor Series for Holomorphic Functions. In Real Analysis, the Taylor series of a given function $f : \mathbb{R} \to \mathbb{R}$ is given by:

$$f(x_0) + f'(x_0)\,(x - x_0) + \frac{f''(x_0)}{2!}\,(x - x_0)^2 + \frac{f'''(x_0)}{3!}\,(x - x_0)^3 + \ldots$$

We have examined some convergence issues and applications of Taylor series in MATH 2033/2043. We also learned that even if the function f is infinitely differentiable everywhere on $\mathbb{R}$, its Taylor series may not converge to that function. In contrast, there is no such an issue in Complex Analysis: as long as the function $f : \mathbb{C} \to \mathbb{C}$ is holomorphic on an open ball $B_\delta(z_0)$, we can show the Taylor series of f:

$$f(z_0) + f'(z_0)\,(z - z_0) + \frac{f''(z_0)}{2!}\,(z - z_0)^2 + \frac{f'''(z_0)}{3!}\,(z - z_0)^3 + \ldots$$

converges pointwise to $f(z)$ on $B_\delta(z_0)$, and uniformly on any smaller ball. As we shall see, it thanks to Cauchy's integral formula. Moreover, the proof of Taylor Theorem in Complex Analysis is also much easier than that in Real Analysis, again thanks to Cauchy's integral formula.

In this chapter, it is more convenient to re-label the variables in the Cauchy's integral formula:

$$f^{(n)}(\alpha) = \frac{n!}{2\pi i} \oint_\gamma \frac{f(z)}{(z - \alpha)^{n+1}}\,dz \quad \longrightarrow \quad f^{(n)}(z) = \frac{n!}{2\pi i} \oint_\gamma \frac{f(\xi)}{(\xi - z)^{n+1}}\,d\xi.$$

For the *re-labelled* Cauchy's integral formula, we require the point z to be enclosed by the simple closed curve γ.

Theorem 4.1 (Taylor Theorem for Holomorphic Functions). *Given a complex-valued function f which is holomorphic on an open ball $B_R(z_0)$, the series:*

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!}\,(z - z_0)^n$$

converges (pointwise) to $f(z)$ for any $z \in B_R(z_0)$.

Proof. Given any $z \in B_R(z_0)$, we let $\varepsilon > 0$ be small enough so that $|z - z_0| < R - \varepsilon$. For simplicity, denote $R' = R - \varepsilon$.

By Cauchy's integral formula, for any $z \in B_{R'}(z_0)$, we have:

$$f(z) = \frac{1}{2\pi i} \oint_{|\xi - z_0| = R'} \frac{f(\xi)}{\xi - z}\, d\xi.$$

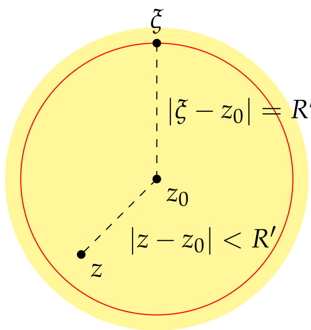
Then, the contour $|z - z_0| = R'$ lies inside the open ball $B_R(z_0)$. The key trick to prove the Taylor Theorem is rewriting $\dfrac{1}{\xi - z}$ as a geometric series. Recall that:

$$\frac{1}{1-w} = 1 + w + w^2 + \ldots \quad \text{whenever } |w| < 1.$$

We first rewrite $\dfrac{1}{\xi - z}$ into this form:

$$\begin{aligned}\frac{1}{\xi - z} &= \frac{1}{(\xi - z_0) - (z - z_0)} = \frac{1}{\xi - z_0} \cdot \frac{1}{1 - \frac{z - z_0}{\xi - z_0}} \\ &= \frac{1}{\xi - z_0} \sum_{n=0}^{\infty} \left(\frac{z - z_0}{\xi - z_0} \right)^n\end{aligned}$$

Here we have used the fact that $\left| \frac{z - z_0}{\xi - z_0} \right| < 1$. See the diagram below. The yellow ball is $B_R(z_0)$, and the red circle is $|\xi - z_0| = R'$.



Then, whenever $z \in B_{R'}(z_0)$, the function $f(z)$ can be expressed as:

$$\begin{aligned}f(z) &= \frac{1}{2\pi i} \oint_{|\xi - z_0| = R'} f(\xi) \cdot \frac{1}{\xi - z}\, d\xi \\ &= \frac{1}{2\pi i} \oint_{|\xi - z_0| = R'} \frac{f(\xi)}{\xi - z_0} \sum_{n=0}^{\infty} \left(\frac{z - z_0}{\xi - z_0} \right)^n d\xi \\ &= \frac{1}{2\pi i} \oint_{|\xi - z_0| = R'} \sum_{n=0}^{\infty} \frac{f(\xi)\,(z - z_0)^n}{(\xi - z_0)^{n+1}}\, d\xi\end{aligned} \tag{4.1}$$

Next we want to see whether we can switch the integral sign $\oint_{|\xi - z_0| = R'}$ and the infinite summation $\sum_{n=0}^{\infty}$. For this we need to show uniform convergence of the series below.

$$\sum_{n=0}^{\infty} \frac{f(\xi)\,(z - z_0)^n}{(\xi - z_0)^{n+1}}.$$

We use Weiestrass's M-test: for any ξ on the circle $\{|\xi - z_0| = R'\}$, we have:

$$\begin{aligned}
\left|\frac{f(\xi)\,(z-z_0)^n}{(\xi-z_0)^{n+1}}\right| &\leq \left|\frac{(z-z_0)^n}{(\xi-z_0)^{n+1}}\right| \underbrace{\sup_{|\xi-z_0|=R'}|f(\xi)|}_{=:C_{R'}} \\
&= \frac{C_{R'}}{R'}\left(\frac{|z-z_0|}{R'}\right)^n
\end{aligned}$$

Since $|z - z_0| < R'$, the series

$$\sum_{n=0}^{\infty} \frac{C_{R'}}{R'}\left(\frac{|z-z_0|}{R'}\right)^n$$

converges. Note that the above series does not depend on ξ (the integration variable). Hence by Weiestrass's M-test, the series $\displaystyle\sum_{n=0}^{\infty} \frac{f(\xi)\,(z-z_0)^n}{(\xi-z_0)^{n+1}}$ converges uniformly on the circle $\{|\xi - z_0| = R'\}$, thus allowing the switch between the integral sign and the summation sign in (4.1):

$$\begin{aligned}
f(z) &= \frac{1}{2\pi i}\sum_{n=0}^{\infty}\oint_{|\xi-z_0|=R'} \frac{f(\xi)\,(z-z_0)^n}{(\xi-z_0)^{n+1}}\,d\xi \\
&= \sum_{n=0}^{\infty}\frac{1}{2\pi i}\left(\oint_{|\xi-z_0|=R'}\frac{f(\xi)}{(\xi-z_0)^{n+1}}\,d\xi\right)(z-z_0)^n \\
&= \sum_{n=0}^{\infty}\frac{f^{(n)}(z_0)}{n!}(z-z_0)^n.
\end{aligned}$$

In the last step we have used the higher order Cauchy's integral formula. □

Example 4.1. The function $f(z) = \sin z$ is an entire function. By straight-forward computations, its derivatives are given by:

$$\begin{aligned}
f'(z) &= \cos z & f''(z) &= -\sin z \\
f^{(3)}(z) &= -\cos z & f^{(4)}(z) &= \sin z \\
&\vdots & &\vdots
\end{aligned}$$

Inductively, it is easy to deduce that $f^{(2k+1)}(0) = (-1)^k$, and $f^{(2k)}(0) = 0$ for any integer $k \geq 0$. Hence, the Taylor series of f about 0 is given by:

$$\begin{aligned}
f(z) &= \sum_{k=0}^{\infty}\frac{f^{(2k+1)}(0)}{(2k+1)!}z^{2k+1} = \sum_{k=0}^{\infty}\frac{(-1)^k}{(2k+1)!}z^{2k+1} \\
&= z - \frac{z^3}{3!} + \frac{z^5}{5!} - \frac{z^7}{7!} + \ldots
\end{aligned}$$

This series converges to $\sin z$ for any $z \in \mathbb{C}$, because $\sin z$ is entire (i.e. holomorphic on every ball $B_R(0)$).

Example 4.2. Consider the function $f(z) = \mathrm{Log}(z)$ which is holomorphic on $\Omega := \mathbb{C}\backslash\{x + 0i : x \leq 0\}$. Note that we can only apply Theorem 4.1 if the ball $B_R(z_0)$ is contained inside Ω.

Let's take $z_0 = 1$ as an example.

$$\begin{aligned}
f'(z) &= \frac{1}{z} & f'(1) &= 1 \\
f''(z) &= -\frac{1}{z^2} & f''(1) &= -1 \\
f^{(3)}(z) &= \frac{2}{z^3} & f^{(3)}(1) &= 2 \\
f^{(4)}(z) &= -\frac{2\times 3}{z^4} & f^{(4)}(1) &= -2\times 3 \\
&\vdots & &\vdots
\end{aligned}$$

Inductively, we deduce that $f^{(n)}(1) = (-1)^{n-1}\cdot(n-1)!$ for $n \geq 1$.

Therefore, the Taylor series for f about 1 is given by:

$$\begin{aligned}
\mathrm{Log}(z) = \mathrm{Log}(1) + \sum_{n=1}^{\infty} \frac{(-1)^{n-1}\cdot(n-1)!}{n!}(z-1)^n &= \sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n}(z-1)^n \\
&= (z-1) - \frac{1}{2}(z-1)^2 + \frac{1}{3}(z-1)^3 - \frac{1}{4}(z-1)^4 + \dots
\end{aligned}$$

Since f is holomorphic on $B_1(1)$ (but not on any larger ball centered at 1), the above Taylor series converges to $\mathrm{Log}(z)$ on $B_1(1)$.

Example 4.3. The Taylor series for some composite functions, such as e^{z^2}, can be derived by substitution instead of deducing the general n-th derivative of the function. For example:

$$\begin{aligned}
e^z &= 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \frac{z^4}{4!} + \dots \\
e^{z^2} &= 1 + z^2 + \frac{(z^2)^2}{2!} + \frac{(z^2)^3}{3!} + \frac{(z^2)^4}{4!} + \dots \\
&= 1 + z^2 + \frac{z^4}{2!} + \frac{z^6}{3!} + \frac{z^8}{4!} + \dots
\end{aligned}$$

Since the series for e^z converges for any $z \in \mathbb{C}$, the series for e^{z^2} converges for any $z \in \mathbb{C}$ as well.

Similarly, by replacing z by $1-z$ in the Taylor series for $\mathrm{Log}(z)$, we get:

$$\mathrm{Log}(1-z) = -z - \frac{1}{2}z^2 - \frac{1}{3}z^3 - \frac{1}{4}z^4 - \dots$$

The series for $\mathrm{Log}(z)$ about 1 converges when $|z-1| < 1$, and so the above series for $\mathrm{Log}(1-z)$ converges when $|(1-z)-1| < 1$, i.e. $|z| < 1$.

Apart from using Theorem 4.1 to find the Taylor series of a given holomorphic function, we can also make use of the geometric series formula directly:

$$\frac{1}{1-w} = 1 + w + w^2 + \dots \quad \text{where } |w| < 1.$$

This method is particularly useful for functions whose n-th derivatives are tedious to compute.

Example 4.4. Consider the function:

$$f(z) = \frac{z-2}{(z+2)(z+3)}.$$

We are going to derive its Taylor series about 0. First, we do partial fractions on the function:

$$f(z) = \frac{5}{z+3} - \frac{4}{z+2}.$$

Then, we try to rewrite each term above in the form of $\frac{a}{1-w}$. Note that:

$$\begin{aligned}
\frac{5}{z+3} &= \frac{5}{3}\cdot\frac{1}{\frac{z}{3}+1} = \frac{5}{3}\cdot\frac{1}{1-(-\frac{z}{3})} \\
&= \frac{5}{3}\sum_{n=0}^{\infty}\left(-\frac{z}{3}\right)^n = \sum_{n=0}^{\infty}(-1)^n\frac{5}{3^{n+1}}z^n && \text{(where } |z|<3\text{)} \\
\frac{4}{z+2} &= \frac{4}{2}\cdot\frac{1}{\frac{z}{2}+1} = \frac{2}{1-(-\frac{z}{2})} \\
&= 2\sum_{n=0}^{\infty}\left(-\frac{z}{2}\right)^n && \text{(where } |z|<2\text{)} \\
&= \sum_{n=0}^{\infty}\frac{(-1)^n}{2^{n-1}}z^n.
\end{aligned}$$

Hence, for $|z|<2$, we have:

$$f(z) = \sum_{n=0}^{\infty}(-1)^n\frac{5}{3^{n+1}}z^n - \sum_{n=0}^{\infty}\frac{(-1)^n}{2^{n-1}}z^n = \sum_{n=0}^{\infty}(-1)^n\left(\frac{5}{3^{n+1}}-\frac{1}{2^{n-1}}\right)z^n.$$

To derive the Taylor series of f about other center (say 1), we can express $\frac{5}{z+3}$ and $\frac{4}{z+2}$ into:

$$\begin{aligned}
\frac{5}{z+3} &= \frac{5}{(z-1)+4} = \frac{5}{4}\cdot\frac{1}{1-\left(-\frac{z-1}{4}\right)} \\
&= \sum_{n=0}^{\infty}\frac{5}{4}\left(-\frac{z-1}{4}\right)^n && \text{(where } |z-1|<4\text{)} \\
&= \sum_{n=0}^{\infty}\frac{(-1)^n5}{4^{n+1}}(z-1)^n \\
\frac{4}{z+2} &= \frac{4}{(z-1)+3} = \frac{4}{3}\cdot\frac{1}{1-\left(-\frac{z-1}{3}\right)} \\
&= \sum_{n=0}^{\infty}\frac{4}{3}\left(-\frac{z-1}{3}\right)^n && \text{(where } |z-1|<3\text{)} \\
&= \sum_{n=0}^{\infty}\frac{(-1)^n4}{3^{n+1}}(z-1)^n.
\end{aligned}$$

Therefore, on $|z-1|<3$, we have:

$$f(z) = \sum_{n=0}^{\infty}(-1)^n\left(\frac{5}{4^{n+1}}-\frac{4}{3^{n+1}}\right)(z-1)^n.$$

Exercise 4.1. Derive the Taylor series of each function below about the given center z_0:

(a) $f(z) = \sin 2z; \quad z_0 = \frac{2\pi}{3}$

(b) $f(z) = \cos 3z; \quad z_0 = \pi$

(c) $f(z) = e^{-z^3}; \quad z_0 = 0$

(d) $f(z) = \operatorname{Log}(3 - 2z); \quad z_0 = 1$

Exercise 4.2. Find the Taylor series about 0 of the functions below up to the z^4 term:

(a) $f(z) = e^{-z} \cos z$

(b) $f(z) = \operatorname{Log}(1 - e^z)$

Exercise 4.3. Find the Taylor series about z_0 of the function below without using Theorem 4.1. State its radius of convergence.

(a) $f(z) = \dfrac{1}{(z-1)(z-2)}, \quad z_0 = 0$

(b) $f(z) = \dfrac{1}{(z-1)(z-2)}, \quad z_0 = i$

Exercise 4.4. Let α, β and z_0 be three distinct complex numbers. Consider the function

$$f(z) = \frac{1}{(z-\alpha)(z-\beta)}.$$

Find the Taylor series about z_0 of the above function, and state its radius of convergence.

Exercise 4.5. Let α be a fixed non-zero complex number. Consider the principal branch of $(1+z)^\alpha$:

$$(1+z)^\alpha := e^{\alpha \operatorname{Log}(1+z)}.$$

Show that its Taylor series about 0 is given by:

$$(1+z)^\alpha = 1 + \sum_{n=1}^{\infty} \frac{\alpha(\alpha-1)\cdot(\alpha-n+1)}{n!} z^n.$$

State its radius of convergence.

4.1.2. Taylor Series with Remainder Term. In Real Analysis, the Taylor Theorem with a remainder term asserts that for any smooth (C^∞) function $f : \mathbb{R} \to \mathbb{R}$, we have:

$$f(x) = \sum_{n=0}^{N-1} \frac{f^{(n)}(a)}{n!}(x-a)^n + \underbrace{\frac{1}{(N-1)!}\int_a^x (x-t)^{N-1} f^{(N)}(t)\, dt}_{=:R_N(x)}$$

The last integral term, commonly denoted as $R_N(x)$, measures how fast the Taylor series converges to $f(x)$ as $N \to \infty$. If $\lim\limits_{N\to\infty} R_N(x) \to 0$ for any x in an interval I, then the Taylor series $\sum\limits_{n=0}^{\infty} \dfrac{f^{(n)}(a)}{n!}(x-a)^n$ converges (pointwise) to $f(x)$ for any $x \in I$. If furthermore, we have:

$$\lim_{N\to\infty} \sup_{x\in I} |R_N(x)| \to 0,$$

then the Taylor series converges uniformly to f on I. However, it is often not easy to show $R_N \to 0$ as the N-th derivative $f^{(N)}$ may not be easy to find.

Back to Complex Analysis, we will soon derive the remainder term for the Taylor series for holomorphic functions. One good thing about the complex version is that the remainder involves only f, but not its derivatives, making it much easier to handle the convergence issue of complex Taylor series. It again thanks to Cauchy's integral formula.

Proposition 4.2. *Let f be a holomorphic function defined on $B_R(z_0)$, then for any $z \in B_R(z_0)$, and any simple closed curve γ in $B_R(z_0)$ enclosing both z and z_0, we have:*

$$f(z) = \sum_{n=0}^{N-1} \frac{f^{(n)}(z_0)}{n!}(z-z_0)^n + \underbrace{\frac{1}{2\pi i}\oint_\gamma \frac{f(\xi)}{\xi - z}\left(\frac{z-z_0}{\xi-z_0}\right)^N d\xi}_{=:R_N(z)}$$

Proof. We only outline the proof since it is modified from the proof of Theorem 4.1. Using Cauchy's integral formula, we first have:

$$f(z) = \frac{1}{2\pi i}\oint_\gamma \frac{f(\xi)}{\xi - z}\, d\xi.$$

The key step in the proof of Theorem 4.1 is to write:

$$\frac{1}{\xi - z} = \frac{1}{(\xi - z_0) - (z - z_0)} = \frac{1}{\xi - z_0} \cdot \frac{1}{1 - \frac{z-z_0}{\xi - z_0}},$$

so that when $\left|\frac{z-z_0}{\xi-z_0}\right| < 1$, we have:

$$\frac{1}{1 - \frac{z-z_0}{\xi-z_0}} = \sum_{n=0}^{\infty}\left(\frac{z-z_0}{\xi-z_0}\right)^n.$$

Now, to prove this proposition, we modify the above key step a bit, by considering:

$$\frac{1 - \left(\frac{z-z_0}{\xi-z_0}\right)^N}{1 - \frac{z-z_0}{\xi-z_0}} = \sum_{n=0}^{N-1}\left(\frac{z-z_0}{\xi-z_0}\right)^n.$$

We leave the rest of the proof for readers (which is a good exercise to test your understanding of the proof of Theorem 4.1). □

Exercise 4.6. Complete the proof of Proposition 4.2.

Exercise 4.7. Consider the remainder term in Proposition 4.2:

$$R_N(z) = \frac{1}{2\pi i}\oint_\gamma \frac{f(\xi)}{\xi - z}\left(\frac{z-z_0}{\xi-z_0}\right)^N d\xi.$$

Let γ be the circle $|\xi - z_0| = R'$ such that $|z - z_0| < R' < R$. Show that:

$$|R_N(z)| \le \frac{R'}{R' - |z-z_0|}\left(\frac{|z-z_0|}{R'}\right)^N \sup_{|\xi - z_0| = R'} |f(\xi)|.$$

Exercise 4.8. Let f be a holomorphic function on $B_R(z_0)$. Using this estimate obtained in Exercise 4.7, deduce that the Taylor series $\displaystyle\sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!}(z-z_0)^n$ converges uniformly to $f(z)$ on any smaller ball $B_r(z_0)$ where $0 < r < R$.

Remark 4.3. The uniform convergence of $\displaystyle\sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!}(z-z_0)^n$ has many remarkable consequences as discussed in MATH 3033/3043. For instance, one can integrate a Taylor series term-by-term.

Exercise 4.9. Consider the Taylor series for $-\mathrm{Log}(1-\xi)$:

$$-\mathrm{Log}(1-\xi) = \xi + \frac{\xi^2}{2} + \frac{\xi^3}{3} + \cdots + \frac{\xi^n}{n} + \cdots \quad \text{where } |\xi| < 1$$

Show that:

$$\frac{z^2}{2} + \frac{z^4}{3 \times 4} + \cdots + \frac{z^{n+1}}{n(n+1)} + \cdots = (1-z)\mathrm{Log}(1-z) + z$$

for any $z \in B_1(0)$.

Exercise 4.10. Show that for any $z \in \mathbb{C}$, we have:

$$\int_0^z e^{-\xi^2}\, d\xi = \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n+1}}{n!(2n+1)}.$$

4.2. Laurent Series

A Laurent series is a "power series" with negative powers of $z - z_0$ as well. The general form of a Laurent series about z_0 is:

$$\begin{aligned}&\sum_{n=1}^{\infty} \frac{a_{-n}}{(z-z_0)^n} + \sum_{n=0}^{\infty} a_n (z-z_0)^n \\ &= \cdots + \frac{a_{-2}}{(z-z_0)^2} + \frac{a_{-1}}{z-z_0} + a_0 + a_1(z-z_0) + a_2(z-z_0)^2 + \cdots\end{aligned}$$

which can be abbreviated as:

$$\sum_{n=-\infty}^{\infty} a_n (z-z_0)^n.$$

A Laurent series is said to be convergent if both $\sum_{n=1}^{\infty} \frac{a_{-n}}{(z-z_0)^n}$ and $\sum_{n=0}^{\infty} a_n (z-z_0)^n$ converge.

If $a_{-n} = 0$ for any negative $-n$, then the Laurent series is a Taylor series. On the other hand, if $a_{-n} \neq 0$ for some negative $-n$, then the Laurent series is undefined when $z = z_0$. As such, a Laurent series is usually defined on an *annular* region $\{r < |z - z_0| < R\}$ instead of a ball centered at z_0. From now on, we denote such an annular region by:

$$A_{R,r}(z_0) := \{z \in \mathbb{C} : r < |z - z_0| < R\}$$

where $R, r \in [0, \infty]$. Note that:

$$\begin{aligned}A_{R,0}(z_0) &= B_R(z_0) \backslash \{z_0\} \\ A_{\infty,r}(z_0) &= \mathbb{C} \backslash \overline{B_r(z_0)} \quad \text{for } r > 0 \\ A_{\infty,0}(z_0) &= \mathbb{C} \backslash \{z_0\}.\end{aligned}$$

4.2.1. Examples of Laurent Series. While a Taylor series gives an analytic expression for a holomorphic function on a *ball*, a Laurent series gives an analytic expression for a function that has a singularity at the center of a ball. Before we discuss a general theorem about Laurent series, let's first look at some examples of writing a function as a Laurent series:

Example 4.5. Consider the function $f : \mathbb{C} \backslash \{1, 2\} \to \mathbb{C}$ defined by:

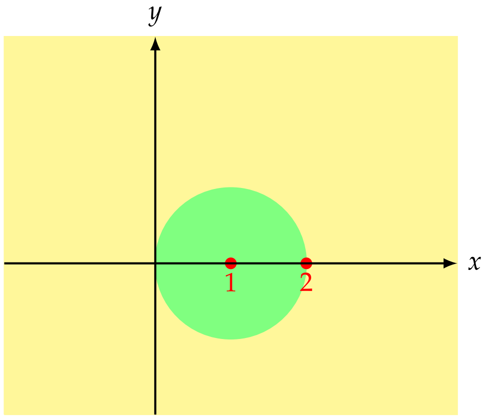
$$f(z) = \frac{1}{(z-1)(z-2)}.$$

It is holomorphic on its domain $\mathbb{C} \backslash \{1, 2\}$. Let's express the above function as a Laurent series about 1:

$$\begin{aligned}\frac{1}{z-2} &= \frac{1}{(z-1)-1} = -\frac{1}{1-(z-1)} \\ &= -\sum_{n=0}^{\infty} (z-1)^n \qquad \text{where } |z-1| < 1.\end{aligned}$$

Hence, on $\in A_{1,0}(1)$, i.e. the green annulus in the figure below, we have:

$$\begin{aligned}f(z) &= \frac{1}{z-1} \cdot \frac{1}{z-2} = -\frac{1}{z-1} \sum_{n=0}^{\infty} (z-1)^n = -\sum_{n=0}^{\infty} (z-1)^{n-1} \\ &= -\frac{1}{z-1} - 1 - (z-1) - (z-1)^2 + \cdots.\end{aligned}$$



However, the green annulus $A_{1,0}(1)$ is not the only annulus centered at 1 on which f is holomorphic. There is another one $A_{\infty,1}(1) = \{1 < |z-1|\}$ centered at 1, i.e. the yellow annulus in the above figure, on which f is also holomorphic. It is also possible to express f as a Laurent series on this yellow annulus:

$$\begin{aligned}\frac{1}{z-2} = \frac{1}{(z-1)-1} &= \frac{1}{z-1} \cdot \frac{1}{1-\frac{1}{z-1}} \\ &= \frac{1}{z-1}\sum_{n=0}^{\infty}\left(\frac{1}{z-1}\right)^n \qquad \text{(where } |z-1|>1\text{)} \\ &= \sum_{n=0}^{\infty}\frac{1}{(z-1)^{n+1}}\end{aligned}$$

Hence, on the yellow annulus $A_{\infty,1}(1)$, the function f can be expressed as the following Laurent series:

$$f(z) = \frac{1}{(z-1)(z-2)} = \frac{1}{z-1}\cdot\frac{1}{z-2} = \sum_{n=0}^{\infty}\frac{1}{(z-1)^{n+2}}.$$

Example 4.6. Find the Laurent series about 0 of the function:

$$f(z) = z^2 e^{\frac{1}{z}}$$

defined on $\mathbb{C}\backslash\{0\}$.

Solution

First recall that the Taylor series for e^w is:

$$e^w = \sum_{n=0}^{\infty}\frac{w^n}{n!} \quad \text{for any } w \in \mathbb{C}.$$

Substitute $w = \frac{1}{z}$, where $z \neq 0$, we get:

$$e^{\frac{1}{z}} = \sum_{n=0}^{\infty}\frac{1}{n!z^n},$$

and hence:

$$\begin{aligned} f(z) &= z^2 e^{\frac{1}{z}} \\ &= z^2 \sum_{n=0}^{\infty} \frac{1}{n! z^n} \\ &= \sum_{n=0}^{\infty} \frac{1}{n! z^{n-2}} \\ &= z^2 + z + \frac{1}{2} + \frac{1}{3! z} + \frac{1}{4! z^2} + \cdots \end{aligned}$$

Exercise 4.11. Express the function:

$$f(z) = \frac{1}{z(z-1)(z-2)}$$

as a Laurent series about 0 in each of the following annuli:

$$A_{1,0}(0), \quad A_{2,1}(0), \quad A_{\infty,2}(0).$$

Also, express the function as a Laurent series about 1 in each of the following annuli:

$$A_{1,0}(1), \quad A_{\infty,1}(1).$$

[Hint: First expand f into partial fractions.]

Exercise 4.12. Find all possible Laurent (or Taylor) series about 1 for the function:

$$f(z) = \frac{1}{z^2 - 2z}.$$

For each series, state the annulus or ball on which it converges.

Exercise 4.13. Find all possible Laurent (or Taylor) series about each z_0 below for the function $f(z) = \frac{1}{z}$.

(a) $z_0 = 0$

(b) $z_0 = 1$

(c) $z_0 = i$

For each series, state the annulus or ball on which it converges.

Exercise 4.14. Show that for any w such that $|w| < 1$, we have:

$$\frac{1}{(1+w)^3} = \sum_{n=2}^{\infty} (-1)^n \frac{n(n-1)}{2} w^{n-2}.$$

[Hint: use Exercise 4.5]

Hence, find all possible Laurent or Taylor series about i for the function:

$$f(z) = \frac{1}{z^3}.$$

For each series, state the annulus or ball on which it converges.

Exercise 4.15. Find the Laurent series about 1 on the annulus $A_{\infty,0}(1)$ for the functions:

$$f(z) = \sin\frac{1}{z-1} \quad \text{and} \quad g(z) = \cos\frac{1}{z-1}.$$

Hence, find the Laurent series about 1 on $A_{\infty,0}(1)$ for:

$$h(z) = \sin\frac{z}{z-1}.$$

Exercise 4.16. What's wrong with the following argument?

$$\frac{z}{1-z} = z\sum_{n=0}^{\infty} z^n = z + z^2 + z^3 + \cdots$$

$$\frac{z}{1-z} = -\frac{1}{1-\frac{1}{z}} = -\sum_{n=0}^{\infty}\frac{1}{z^n} = -1 - \frac{1}{z} - \frac{1}{z^2} - \cdots$$

By subtraction, we get:

$$0 = \cdots + \frac{1}{z^2} + \frac{1}{z} + 1 + z + z^2 + \cdots = \sum_{n=-\infty}^{\infty} z^n.$$

4.2.2. Existence Theorem of Laurent Series. We have learned how to express a function into a Laurent series through examples. Next, we proved a general existence theorem of Laurent series for *any* holomorphic function on *any* annular region.

Theorem 4.4 (Laurent Theorem). *Let f be a holomorphic function defined on an annulus $A_{R,r}(z_0) := \{r < |z - z_0| < R\}$ where $R, r \in [0, \infty]$, then f can be expressed as a Laurent series about z_0 on the annulus $A_{R,r}(z_0)$:*

$$f(z) = \sum_{n=-\infty}^{\infty} c_n (z - z_0)^n$$

for some complex numbers c_n's.

Proof. The proof is similar to that of Taylor's series, but is a bit trickier since an annulus is not simply-connected and so Cauchy's integral formula cannot be applied directly.

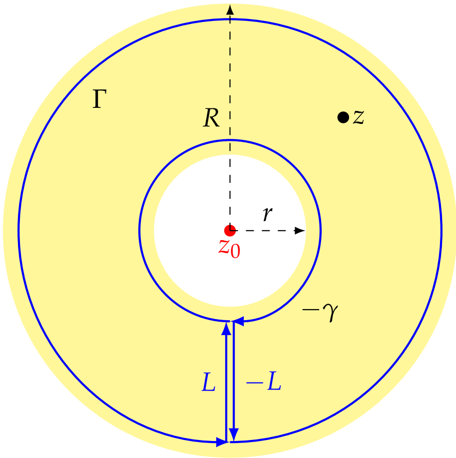
Fix $z \in A_{R,r}(z_0)$, we first consider a simple closed curve Γ in $A_{R,r}(z_0)$ which encloses both z and z_0 (just like in the proof of Taylor's Theorem). However, we cannot apply Cauchy's integral formula on the integral:

$$\frac{1}{2\pi i}\oint_{\Gamma} \frac{f(\xi)}{\xi - z_0}\, d\xi$$

since f is not holomorphic on $B_R(z_0)$. However, we can construct a "key-hole" contour:

$$C = \Gamma + L - \gamma - L$$

where $-\gamma$ is the clockwise circle, and L is a straight-path as shown in the figure below. We can pick Γ to be the circle with radius slightly smaller than R, and γ with radius slightly bigger than r so that C encloses z.



Under such a construction, the contour $C = \Gamma + L - \gamma - L$ is a simple closed curve and the region enclosed by C becomes simply connected. We can then apply Cauchy's integral formula:

$$
\begin{aligned}
f(z) &= \frac{1}{2\pi i} \oint_C \frac{f(\xi)}{\xi - z} \, d\xi \\
&= \frac{1}{2\pi i} \left(\oint_\Gamma + \oint_L - \oint_\gamma - \oint_L \right) \frac{f(\xi)}{\xi - z} \, d\xi \\
&= \frac{1}{2\pi i} \oint_\Gamma \frac{f(\xi)}{\xi - z} \, d\xi - \frac{1}{2\pi i} \oint_\gamma \frac{f(\xi)}{\xi - z} \, d\xi.
\end{aligned}
\tag{4.2}
$$

The key idea of the proof is to express the integral over Γ as a series of non-negative powers, and the integral over γ as a series of negative powers.

When $\xi \in \Gamma$, we have $|z - z_0| < |\xi - z_0|$, so:

$$\frac{1}{\xi - z} = \frac{1}{(\xi - z_0) - (z - z_0)} = \frac{1}{\xi - z_0} \cdot \frac{1}{1 - \frac{z - z_0}{\xi - z_0}} = \frac{1}{\xi - z_0} \sum_{n=0}^{\infty} \left(\frac{z - z_0}{\xi - z_0} \right)^n$$

Hence, the first integral becomes:

$$\frac{1}{2\pi i} \oint_\Gamma \frac{f(\xi)}{\xi - z} \, d\xi = \frac{1}{2\pi i} \oint_\Gamma \sum_{n=0}^{\infty} \frac{f(\xi)}{\xi - z_0} \left(\frac{z - z_0}{\xi - z_0} \right)^n \, d\xi.$$

In order to switch the infinite summation and the integral sign, we justify that the series converges uniformly on $\xi \in \Gamma$. Suppose Γ has radius R', then for any $\xi \in \Gamma$:

$$\left| \frac{f(\xi)}{\xi - z_0} \left(\frac{z - z_0}{\xi - z_0} \right)^n \right| \leq \frac{1}{R'} \left(\frac{|z - z_0|}{R'} \right)^n \sup_\Gamma |f| \, .$$

Note that $\sup_\Gamma |f|$ is finite by compactness of Γ. Since $|z - z_0| < R'$, the geometric series

$$\sum_{n=0}^{\infty} \frac{1}{R'} \left(\frac{|z - z_0|}{R'} \right)^n \sup_\Gamma |f|$$

converges. By Weierstrass's M-test, the series

$$\sum_{n=0}^{\infty} \frac{f(\xi)}{\xi - z_0} \left(\frac{z - z_0}{\xi - z_0} \right)^n$$

converges uniformly on $\xi \in \Gamma$, so one can switch the summation and integral signs and get:

$$\frac{1}{2\pi i}\oint_\Gamma \frac{f(\xi)}{\xi - z}\,d\xi = \sum_{n=0}^{\infty}\left(\frac{1}{2\pi i}\oint_\Gamma \frac{f(\xi)}{(\xi - z_0)^{n+1}}\,d\xi\right)(z - z_0)^n. \tag{4.3}$$

The second integral can be handled similarly. The difference is that when $\xi \in \gamma$, we have $|\xi - z_0| < |z - z_0|$ instead. We instead write:

$$\frac{1}{\xi - z} = \frac{1}{(\xi - z_0) - (z - z_0)} = -\frac{1}{z - z_0}\cdot\frac{1}{1 - \frac{\xi - z_0}{z - z_0}} = -\frac{1}{z - z_0}\sum_{n=0}^{\infty}\left(\frac{\xi - z_0}{z - z_0}\right)^n$$

Hence,

$$\frac{1}{2\pi i}\oint_\gamma \frac{f(\xi)}{\xi - z}\,d\xi = -\frac{1}{2\pi i}\oint_\gamma \sum_{n=0}^{\infty}\frac{f(\xi)}{z - z_0}\left(\frac{\xi - z_0}{z - z_0}\right)^n d\xi.$$

We leave it as an exercise for readers to argue that the series converges uniformly on $\xi \in \gamma$ so that we can switch the integral and summations signs:

$$\frac{1}{2\pi i}\oint_\gamma \frac{f(\xi)}{\xi - z}\,d\xi = -\sum_{n=0}^{\infty}\left(\frac{1}{2\pi i}\oint_\gamma f(\xi)(\xi - z_0)^n\,d\xi\right)\frac{1}{(z - z_0)^{n+1}}. \tag{4.4}$$

Combining (4.2), (4.3) and (4.4), we obtain:

$$\begin{aligned} f(z) = &\underbrace{\sum_{n=1}^{\infty}\left(\frac{1}{2\pi i}\oint_\gamma f(\xi)(\xi - z_0)^{n-1}\,d\xi\right)\frac{1}{(z - z_0)^n}}_{(4.4)} \\ &+ \underbrace{\sum_{n=0}^{\infty}\left(\frac{1}{2\pi i}\oint_\Gamma \frac{f(\xi)}{(\xi - z_0)^{n+1}}\,d\xi\right)(z - z_0)^n}_{(4.3)} \end{aligned}$$

It completes the proof by defining

$$\begin{aligned} c_{-n} &= \frac{1}{2\pi i}\oint_\gamma f(\xi)(\xi - z_0)^{n-1}\,d\xi && \text{for } -n = -1, -2, -3, \cdots \\ c_n &= \frac{1}{2\pi i}\oint_\Gamma \frac{f(\xi)}{(\xi - z_0)^{n+1}}\,d\xi && \text{for } n = 0, 1, 2, 3, \cdots \end{aligned}$$

□

Exercise 4.17. Justify the claim in the above proof that the series:

$$\sum_{n=0}^{\infty}\frac{f(\xi)}{z - z_0}\left(\frac{\xi - z_0}{z - z_0}\right)^n$$

converges uniformly on $\xi \in \gamma$ (and z, z_0 are considered to be fixed).

Remark 4.5. Although from the proof of Theorem 4.4 one can express the coefficient c_n's of a Laurent series in terms of contour integrals, we do not usually find the coefficients this way since these contour integrals may not be easy to compute.

4.2.3. Laurent Series with Remainders. Similar to Taylor series, one can refine Theorem 4.4 a bit by deriving the remainder terms. Using the remainder terms, one can argue that for a holomorphic function f defined on an annulus $A_{R,r}(z_0)$, the Laurent series converges *uniformly* to f on every smaller annulus $A_{R',r'}(z_0)$ (where $r < r' < R' < R$). This result is remarkable as it allows us to integrate a Laurent's series term-by-term.

Proposition 4.6. *Let f be a holomorphic function on the annulus $A_{R,r}(z_0)$, where $0 \leq r < R \leq \infty$. Then, for each positive integer N and $z \in A_{R,r}(z_0)$, we have:*

$$f(z) = \sum_{n=1}^{N} \left(\frac{1}{2\pi i} \oint_{\gamma} f(\xi)(\xi - z_0)^{n-1}\, d\xi \right) \frac{1}{(z-z_0)^n} + \underbrace{\frac{1}{2\pi i} \oint_{\gamma} \frac{f(\xi)}{z-\xi} \left(\frac{\xi - z_0}{z - z_0} \right)^N d\xi}_{=:r_N(z)}$$

$$+ \sum_{n=0}^{N-1} \left(\frac{1}{2\pi i} \oint_{\Gamma} \frac{f(\xi)}{(\xi - z_0)^{n+1}}\, d\xi \right) (z-z_0)^n + \underbrace{\frac{1}{2\pi i} \oint_{\Gamma} \frac{f(\xi)}{\xi - z} \left(\frac{z - z_0}{\xi - z_0} \right)^N d\xi}_{=:R_N(z)}$$

where Γ and γ are any pair of circles in $A_{R,r}(z_0)$ centered at z_0 such that z is bounded between Γ and γ.

Proof. We leave the proof of Proposition 4.6 as an exercise. It is very similar to the proof of Proposition 4.2 for Taylor series. Readers should first digest the whole proof of Proposition 4.2, then write up a coherent proof for this proposition. □

Exercise 4.18. Prove Proposition 4.6. Using this, show that the Laurent series about z_0 for f converges uniformly on every smaller annulus $A_{R',r'}(z_0)$ where $r < r' < R' < R$. [Hint: show that both remainders $R_N(z)$ and $r_N(z)$ converge uniformly to 0 on $A_{R',r'}(z_0)$ as $N \to \infty$.]

One practical use of uniform convergence is term-by-term integration. For example, consider the function $f(z) = z^2 e^{\frac{1}{z}}$, which can be expressed as a Laurent series:

$$z^2 e^{\frac{1}{z}} = z^2 + z + \frac{1}{2} + \frac{1}{3!z} + \frac{1}{4!z^2} + \cdots$$

Then, to integrate $f(z)$ over the circle $|z| = 1$, we can integrate the Laurent series term-by-term:

$$\begin{aligned}
&\oint_{|z|=1} z^2 e^{\frac{1}{z}}\, dz \\
&= \oint_{|z|=1} z^2\, dz + \oint_{|z|=1} z\, dz + \oint_{|z|=1} \frac{1}{2}\, dz + \oint_{|z|=1} \frac{1}{3!z}\, dz + \oint_{|z|=1} \frac{1}{4!z^2}\, dz + \cdots \\
&= 0 + 0 + 0 + \frac{2\pi i}{3!} + 0 + 0 + \cdots = \frac{\pi i}{6}.
\end{aligned}$$

Recall that for any simple closed γ enclosing the origin, the contour integral

$$\oint_{\gamma} z^n\, dz$$

is non-zero *only* when $n = -1$.

From the above example, we see the significance of expressing a function as a Laurent series. To compute a contour integral, it often amounts to finding the coefficient c_{-1} of the Laurent series. It leads to the develop of residue theory to be discussed in the next section.

4.3. Residue Calculus

In this section we discuss both theory and applications of an important topic in Complex Analysis: residue calculus. It has many powerful applications on evaluations of some complicated *real* integrals that physicists and engineers often encounter.

4.3.1. Classification of Singularities. A singular point, or singularity, refers to a point z_0 at which a function f fails to be complex differentiable. For instance, 1 and 2 are singularities of the function:

$$f(z) = \frac{1}{(z-1)(z-2)}.$$

It is possible for a function to have infinitely many singularities, such as:

$$g(z) = \frac{1}{\sin z}$$

whose singularities are 0, $\pm\pi$, $\pm 2\pi$, etc.

Some functions even have singularities that form a "cluster". For instance, consider:

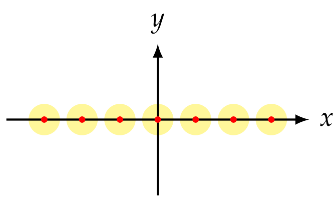
$$h(z) = \frac{1}{\sin \frac{1}{z}}$$

which is singular when $z \in \{\frac{\pi}{n} : n \in \mathbb{Z}\} \cup \{0\}$. The singular set $\{\frac{\pi}{n} : n \in \mathbb{Z}\} \cup \{0\}$ form a cluster around 0, meaning there is no way to find an annulus $A_{R,0}(0)$ centered at 0 such that h is holomorphic on $A_{R,0}(0)$. Hence, it is not possible to analyze the function h by a Laurent series about 0 on $A_{R,0}(0)$.

In order to utilize Laurent series, we focus on those singularities that can be *isolated* from others. We have the following terminology:

> **Definition 4.7** (Isolated Singularity)**.** A point z_0 is said to be an *isolated singularity* for a function $f(z)$ if there exists $\varepsilon > 0$ such that f is holomorphic on $A_{\varepsilon,0}(z_0) = B_\varepsilon(z_0)\backslash\{z_0\}$.

For the function $g(z) = \dfrac{1}{\sin z}$, all singularities are isolated as depicted in the diagram below:



Around every isolated singularity z_0 of a function $f(z)$, it is possible (thanks to Theorem 4.4) to express the function f as a Laurent series on a small annulus $A_{\varepsilon,0}(z_0)$:

$$f(z) = \sum_{n=-\infty}^{\infty} c_n (z-z_0)^n.$$

Depending on the *smallest* n such that $c_n \neq 0$, we have the following terminology:

- If $c_{-1} = c_{-2} = c_{-3} = \cdots = 0$, then z_0 is said to be a *removable singularity* of f. For instance, 0 is such a singularity for the function:

$$\frac{\sin z}{z} = 1 - \frac{z^2}{3!} + \frac{z^4}{5!} - \frac{z^6}{7!} + \cdots$$

- If k is a positive integer such that $c_{-k} \neq 0$ while $c_{-(k+1)} = c_{-(k+2)} = \cdots = 0$, then z_0 is said to be a *pole of order* k of f. For instance, 0 is a pole of order 3 for the function:
$$\frac{\sin z}{z^4} = \frac{1}{z^3} - \frac{1}{3!z} + \frac{z}{5!} - \frac{z^3}{7!} + \cdots$$
Moreover, a pole of order 1 is usually called a *simple pole*.
- If $c_{-n} \neq 0$ for infinitely many negative integers $-n$, then z_0 is said to be an *essential singularity*. For instance, 0 is such a singularity for the function:
$$e^{\frac{1}{z}} = 1 + \frac{1}{z} + \frac{1}{2!z^2} + \frac{1}{3!z^3} + \cdots$$

If z_0 is a removable singularity for $f : B_R(z_0)\backslash\{z_0\} \to \mathbb{C}$, then one can define $f(z_0) := c_0$ so that f extends to become a holomorphic function on $B_R(z_0)$. That's why we can z_0 removable. Similarly, if z_0 is a pole of order n for $f : A_{R,0}(z_0) \to \mathbb{C}$, then $(z - z_0)^n f(z)$ extends to become a holomorphic function on $B_R(z_0)$. However, a function with an essential singularity cannot be extended to become a holomorphic function in a similar way (that's why we call it essential).

To determine the order of a pole, we may simply find its Laurent series expansion. However, sometimes it is not easy to do so, such as 0 for the function $\dfrac{1}{\sin z}$. An alternative way to find the order of a pole is to consider the limit:
$$\lim_{z \to z_0} (z - z_0)^k f(z).$$
If k is an integer such that:
$$\lim_{z \to z_0} (z - z_0)^k f(z) \text{ exists and is non-zero,}$$
then the order of the pole z_0 is k. For example, since:
$$\lim_{z \to 0} \frac{z}{\sin z} = 1 \neq 0,$$
0 is a pole of order 1 for the function $\dfrac{1}{\sin z}$. Hence, one can express this function as a Laurent series on a small annulus $A_{\varepsilon,0}(0)$:
$$\frac{1}{\sin z} = \frac{c_{-1}}{z} + c_0 + c_1 z + c_2 z^2 + \cdots$$
Multiplying z on both sides, we get:
$$\frac{z}{\sin z} = c_{-1} + c_0 z + c_1 z^2 + c_2 z^3 + \cdots$$
and by letting $z \to 0$, we can also conclude that $c_{-1} = 1$. Therefore, if γ is a simple close curve enclosing 0 in this small annulus $A_{\varepsilon,0}(0)$, then we have:
$$\oint_\gamma \frac{1}{\sin z}\, dz = \oint_\gamma \left(\frac{1}{z} + c_0 + c_1 z + c_2 z^2 + \cdots \right) dz = \oint_\gamma \frac{1}{z}\, dz + 0 + 0 + \cdots = 2\pi i.$$

Exercise 4.19. Find all isolated singularities of each function below, and classify the nature of these singularities. For poles, state also their orders.

(a) $f(z) = \dfrac{e^z - 1}{z}$

(b) $g(z) = \dfrac{\text{Log}(z)}{(z-3)^5}$

(c) $h(z) = z^{4023} \cos \frac{1}{z}$

4.3.2. Residues. As illustrated in many examples, the coefficient c_{-1} of a Laurent series plays a special role in evaluating a contour integral. It is special in a sense that for an integer n,

$$\oint_{|z-z_0|=\varepsilon} (z-z_0)^n \, dz = \begin{cases} 2\pi i & \text{if } n = -1 \\ 0 & \text{otherwise} \end{cases}$$

Hence, to integrate a Laurent series, one only needs to integrate the term $\dfrac{c_{-1}}{z-z_0}$, which can be done by Cauchy's integral formula. In view of the special role of c_{-1}, we define:

Definition 4.8 (Residues)**.** Let z_0 be an isolated singularity of $f(z)$ such that the Laurent series about z_0 for f on some annulus $A_{\varepsilon,0}(z_0)$ is given by:

$$f(z) = \sum_{n=-\infty}^{\infty} c_n (z-z_0)^n,$$

then we denote and define the *residue of f at z_0* by:

$$\operatorname{Res}(f, z_0) := c_{-1}.$$

Example 4.7. Find the residue of the function

$$f(z) = \frac{z^2 - 2z}{(z+1)^2(z^2+4)}$$

at each of its isolated singularity.

Solution

The denominator has roots -1, $2i$ and $-2i$, hence they are isolated singularities of f. In this solution, we will decompose $f(z)$ into partial fractions. It may not be a pleasant way finding residues, but we will later provide an easier way.

Note that f is a rational function, we can break it into partial fractions:

$$f(z) = \frac{A}{(z+1)^2} + \frac{B}{z+1} + \frac{C}{z-2i} + \frac{D}{z+2i}.$$

We leave it as an exercise for readers to determine the value of A, B, C and D. One should be able to get:

$$f(z) = \frac{\frac{3}{5}}{(z+1)^2} + \frac{-\frac{14}{15}}{z+1} + \underbrace{\frac{\frac{7+i}{25}}{z-2i} + \frac{\frac{7-i}{25}}{z+2i}}_{\text{holomorphic near } -1}.$$

On a small annulus $A_{\varepsilon,0}(-1)$ about -1, the last two terms $\dfrac{\frac{7+i}{25}}{z-2i} + \dfrac{\frac{7-i}{25}}{z+2i}$ are holomorphic. Therefore, if one express them as a Laurent series about -1, only non-negative powers of $z+1$ will appear, and the coefficient of $\frac{1}{z+1}$ will not be affected. Therefore, we have:

$$\operatorname{Res}(f, -1) = -\frac{14}{15}.$$

By a similar reason, we have:

$$\operatorname{Res}(f, 2i) = \frac{7+i}{25} \text{ and } \operatorname{Res}(f, -2i) = \frac{7-i}{25}.$$

Exercise 4.20. Determine all isolated singularities of the function
$$f(z) = \frac{z^2+1}{(z+1)(z-1)^2}.$$
and find the residue at each isolated singularity.

It is no doubt that partial fraction decompositions are time-consuming and not fun (it may remind you the computational nightmare you might have encountered in MATH 1014). Fortunately, there is a better way for finding residues for *poles* (does not work for essential singularity).

If we know already that z_0 is a pole of order 1 (i.e. simple pole) of a function $f(z)$, then
$$f(z) = \frac{c_{-1}}{z-z_0} + c_0 + c_1(z-z_0) + c_2(z-z_0)^2 + \cdots$$
It is then easy to see that
$$c_{-1} = \lim_{z\to z_0} (z-z_0)f(z).$$
Therefore, in order to find $\text{Res}(f, z_0)$ for a simple pole z_0, we simply need to compute the above limit.

Now consider the case if z_0 is a pole of order k for f, then its Laurent series about z_0 is given by:
$$f(z) = \frac{c_{-k}}{(z-z_0)^k} + \frac{c_{-(k-1)}}{(z-z_0)^{k-1}} + \cdots + \frac{c_{-1}}{z-z_0} + c_0 + c_1(z-z_0) + c_2(z-z_0)^2 + \cdots$$
Our goal is to find c_{-1}. By multiplying both sides by $(z-z_0)^k$, we can get:
$$(z-z_0)^k f(z) = c_{-k} + c_{-(k-1)}(z-z_0) + \cdots + c_{-1}(z-z_0)^{k-1} + c_0(z-z_0)^k + \cdots$$
By differentiating both sides for $k-1$ times, all terms involving $(z-z_0)^n$ with $n < k-1$ will disappear:
$$\frac{d^{k-1}}{dz^{k-1}}(z-z_0)^k f(z) = c_{-1}(k-1)! + \widetilde{c}_0(z-z_0) + \widetilde{c}_1(z-z_0)^2 + \cdots$$
We have used the fact that $\frac{d^{k-1}}{dz^{k-1}}(z-z_0)^{k-1} = (k-1)!$, and $\widetilde{c}_0, \widetilde{c}_1, \ldots$ are some complex numbers (which we do not need to know their values).

By letting $z \to z_0$, we get:
$$\lim_{z\to z_0} \frac{d^{k-1}}{dz^{k-1}}(z-z_0)^k f(z) = c_{-1}(k-1)!$$
which provides a good way to find c_{-1} without expanding a Laurent series:

Proposition 4.9. *Suppose z_0 is a pole of order $k < \infty$ for a function f, then we have:*
$$\text{Res}(f, z_0) = \frac{1}{(k-1)!} \lim_{z\to z_0} \frac{d^{k-1}}{dz^{k-1}}(z-z_0)^k f(z).$$
In particular, for a simple pole z_0, we have:
$$\text{Res}(f, z_0) = \lim_{z\to z_0}(z-z_0)f(z).$$

Proof. See the preceding paragraph. □

Example 4.8. Find the residue of the function

$$f(z) = \frac{z^2 - 2z}{(z+1)^2(z^2+4)}$$

of each isolated singularity using Proposition 4.9.

Solution

As discussed before, the isolated singularities are -1, $2i$ and $-2i$. Observe that:

$$\lim_{z\to -1}(z+1)^2 f(z) = \lim_{z\to -1}\frac{z^2-2z}{z^2+4} = \frac{3}{5} \neq 0.$$

Hence -1 is a pole of order 2. From Proposition 4.9, we have:

$$\begin{aligned}
\operatorname{Res}(f,-1) &= \frac{1}{(2-1)!}\lim_{z\to -1}\frac{d^{2-1}}{dz^{2-1}}(z+1)^2 f(z) \\
&= \lim_{z\to -1}\frac{d}{dz}\frac{z^2-2z}{z^2+4} \\
&= \lim_{z\to -1}\frac{2z^2+8z-8}{(z^2+4)^2} = -\frac{14}{25}.
\end{aligned}$$

Both $2i$ and $-2i$ are simple poles, so we have:

$$\operatorname{Res}(f,2i) = \lim_{z\to 2i}(z-2i)f(z) = \lim_{z\to 2i}\frac{z^2-2z}{(z+1)^2(z+2i)} = \frac{7+i}{25}$$

$$\operatorname{Res}(f,-2i) = \lim_{z\to -2i}(z+2i)f(z) = \lim_{z\to -2i}\frac{z^2-2z}{(z+1)^2(z-2i)} = \frac{7-i}{25}$$

Example 4.9. Find the residue at 0 of each function below:

$$f(z) = \frac{e^z}{\sin z} \qquad g(z) = \frac{e^z - 1}{\sin z} \qquad h(z) = \frac{e^z}{\sin^2 z}$$

Solution

For each function, we first determine whether 0 is a pole, and find out its order.

For $f(z)$, we consider:

$$\lim_{z\to 0} z f(z) = \lim_{z\to 0}\frac{z}{\sin z}\cdot e^z = 1\cdot e^0 = 1 \neq 0.$$

Hence 0 is a simple pole for f, and $\operatorname{Res}(f,0) = 1$.

For $g(z)$, note that:

$$\lim_{z\to 0} g(z) = \lim_{z\to 0}\frac{z + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots}{z - \frac{z^3}{3!} + \frac{z^5}{5!} - \cdots} = \lim_{z\to 0}\frac{1 + \frac{z}{2!} + \frac{z^2}{3!} + \cdots}{1 - \frac{z^2}{3!} + \frac{z^4}{5!} - \cdots} = 1 < \infty.$$

Hence 0 is a removable singularity of $g(z)$, and there is no $\frac{1}{z}$-term in the Laurent series, and so $\operatorname{Res}(g,0) = 0$.

For $h(z)$:

$$\lim_{z\to 0} z^2 h(z) = \lim_{z\to 0}\left(\frac{z}{\sin z}\right)^2 e^z = 1 \neq 0.$$

Hence 0 is a pole of order 2 for h. By Proposition 4.9, we can find:

$$\begin{aligned}
\mathrm{Res}(h,0) &= \frac{1}{1!}\lim_{z\to 0}\frac{d}{dz}z^2h(z) = \lim_{z\to 0}\frac{d}{dz}\frac{z^2e^z}{\sin^2 z} \\
&= \lim_{z\to 0}\frac{\sin^2 z\,(2ze^z+z^2e^z) - z^2e^z\cdot 2\sin z\cos z}{\sin^4 z} \\
&= \lim_{z\to 0}\left[\frac{z^2e^z}{\sin^2 z} + 2ze^z\left(\frac{\sin z - z\cos z}{\sin^3 z}\right)\right] \\
&= 1 + \lim_{z\to 0} 2ze^z\left(\frac{\left(z - \frac{z^3}{3!} + \frac{z^5}{5!} - \cdots\right) - z\left(1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \cdots\right)}{\sin^3 z}\right) \\
&= 1 + \lim_{z\to 0} 2ze^z\left(\frac{(\frac{1}{2!}-\frac{1}{3!})z^3 - (\frac{1}{4!}-\frac{1}{5!})z^5 + \cdots}{\sin^3 z}\right) \\
&= 1 + \lim_{z\to 0} 2ze^z\left(\left(\frac{1}{2!}-\frac{1}{3!}\right)\frac{z^3}{\sin^3 z} - \left(\frac{1}{4!}-\frac{1}{5!}\right)\frac{z^5}{\sin^3 z} + \cdots\right) \\
&= 1 + 0\cdot e^0\cdot\left(\frac{1}{2!}-\frac{1}{3!} + 0 + 0 + \cdots\right) \\
&= 1.
\end{aligned}$$

Exercise 4.21. For each function below, find its residue at each isolated singularity using any method:

$$\frac{z^2-1}{z^3(z^2+1)} \qquad \frac{1}{6z^2+8z+9} \qquad \frac{z^{1997}-1}{z^{2047}-1}$$

$$\frac{1}{e^z-1} \qquad \frac{e^{2zi}}{\sin z} \qquad \frac{e^{2zi}-1}{\sin z}$$

$$\frac{1}{z\sin z} \qquad \frac{z^2}{e^{1/z}} \qquad \frac{\sin z}{z^2(z-\pi)^3}$$

Exercise 4.22. Compute the following residues:

(a) $\mathrm{Res}\left(\dfrac{1}{2\cos z - 2 + z^2}, 0\right)$

(b) $\mathrm{Res}\left(\dfrac{z^{2n}}{(z-1)^n}, 1\right)$

4.3.3. Residue Theorem. The residue $\mathrm{Res}(f, z_0)$ of an isolated singularity z_0 determines the value of a contour integral $\oint_\gamma f(z)\,dz$ where γ is a tiny simple closed curve so that z_0 is the only singularity it encloses. Namely, we have $\oint_\gamma f(z)\,dz = 2\pi i\,\mathrm{Res}(f, z_0)$.

If a simple closed curve γ encloses more than one isolated singularities $\{z_1, \cdots, z_N\}$, then we may first express the contour integral over γ as the sum of contour integrals:

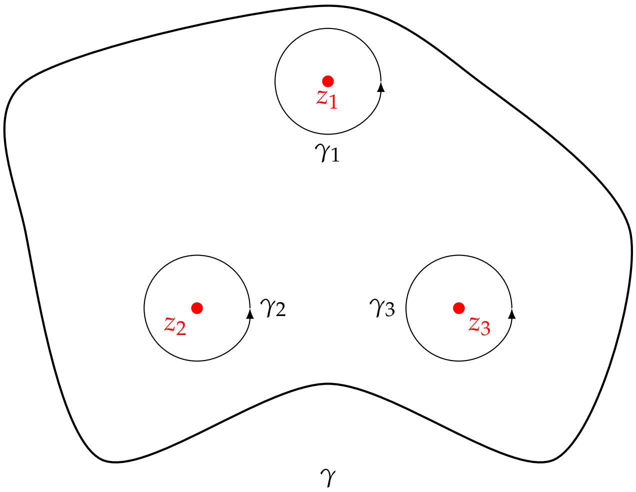
$$\oint_\gamma f(z)\,dz = \oint_{\gamma_1} f(z)\,dz + \cdots + \oint_{\gamma_N} f(z)\,dz$$

where each γ_j is a small simple-closed curve so that z_j is the only singularity it encloses. Then, each γ_j-integral is given by $2\pi i\,\mathrm{Res}(f, z_j)$, and hence we have the following theorem:

Theorem 4.10 (Residue Theorem)**.** *Let $f : \Omega \to \mathbb{C}$ be a complex-valued functions whose singularities are all isolated. Let γ be a simple closed curve, and $z_1, \cdots, z_N \in \Omega$ be all the singularities enclosed by γ. Then, we have:*

$$\oint_\gamma f(z)\,dz = 2\pi i \sum_{j=1}^{N} \operatorname{Res}(f, z_j).$$

Proof. Let $\varepsilon > 0$ be sufficiently such that each circle $\{|z - z_j| = \varepsilon\}$, denoted by γ_j, encloses z_j as the only singularity of f (see figure below).



Then, by the standard hole-drilling argument, we have:

$$\oint_\gamma f(z)\,dz = \oint_{\gamma_1} f(z)\,dz + \cdots + \oint_{\gamma_N} f(z)\,dz$$

Each γ_j encloses z_j as the only singularity of f. Express f as a Laurent series on $A_{\varepsilon,0}(z_j)$:

$$f(z) = \sum_{n=-\infty}^{\infty} c_n (z - z_j)^n.$$

Recall that $\oint_{\gamma_j} (z - z_j)^n\,dz \neq 0$ only when $n = -1$, and by uniform convergence of Laurent series, we get:

$$\oint_{\gamma_j} f(z) = 2\pi i c_{-1} = 2\pi i \operatorname{Res}(f, z_j).$$

Therefore, we have:

$$\oint_\gamma f(z)\,dz = 2\pi i \sum_{j=1}^{N} \operatorname{Res}(f, z_j),$$

completing the proof. $\square$

Example 4.10. Use Residue Theorem to evaluate the contour integral:

$$\oint_{|z|=R} \frac{z^2 - 2z}{(z+1)^2(z^2+4)}\, dz$$

where R is in the range of:

(a) $0 < R < 1$

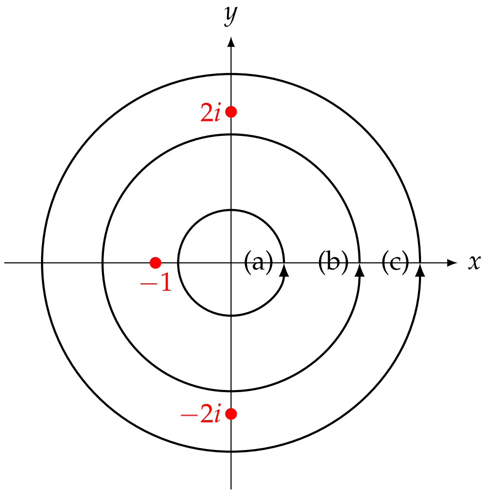
(b) $1 < R < 2$

(c) $2 < R$.

Solution

Denote $f(z) = \dfrac{z^2 - 2z}{(z+1)^2(z^2+4)}$. The singularities of f are -1, $2i$ and $-2i$. We have calculated in Example 4.8 that

$$\operatorname{Res}(f, -1) = -\frac{14}{15} \qquad \operatorname{Res}(f, 2i) = \frac{7+i}{25} \qquad \operatorname{Res}(f, -2i) = \frac{7-i}{25}$$



(a) When $0 < R < 1$, the circle $|z| = R$ does not enclose any singularities, hence

$$\oint_{|z|=R} f(z)\, dz = 0.$$

(b) When $1 < R < 1$, the circle $|z| = R$ encloses the singularity -1 only, hence

$$\oint_{|z|=R} f(z)\, dz = 2\pi i \operatorname{Res}(f, -1) = -\frac{28\pi i}{15}$$

(c) When $R > 2$, the circle $|z| = R$ encloses all three singularities, hence

$$\begin{aligned}\oint_{|z|=R} f(z)\, dz &= 2\pi i\ (\operatorname{Res}(f, -1) + \operatorname{Res}(f, 2i) + \operatorname{Res}(f, -2i)) \\ &= 2\pi i \left(-\frac{14}{15} + \frac{7+i}{25} + \frac{7-i}{25}\right) = -\frac{56\pi i}{75}.\end{aligned}$$

Example 4.11. Let N be a positive integer and γ_N be the square contour with vertices $\pm(N+\frac{1}{2})\pm(N+\frac{1}{2})i$. Use Residue Theorem to show:

$$\sum_{n=1}^{N}\frac{1}{n^2}=\frac{\pi^2}{3}+\frac{1}{2\pi i}\oint_{\gamma_N}\frac{\pi}{z^2}\cot\pi z\,dz.$$

Hence, deduce that:

$$1+\frac{1}{2^2}+\frac{1}{3^2}+\cdots=\frac{\pi^2}{6}.$$

Solution

Denote $f(z):=\dfrac{\pi}{z^2}\cot\pi z=\dfrac{\pi\cos\pi z}{z^2\sin\pi z}$. Its singularities are the set of all integers n.

First we observe that

$$\lim_{z\to 0}z^3f(z)=\lim_{z\to 0}\frac{\pi z}{\sin\pi z}\cdot\cos\pi z=1,$$

so 0 is a pole of order 3 for f. By Proposition 4.9, its residue is given by:

$$\begin{aligned}
\operatorname{Res}(f,0)&=\frac{1}{2!}\lim_{z\to 0}\frac{d^2}{dz^2}z^3f(z)=\frac{1}{2}\lim_{z\to 0}\frac{d^2}{dz^2}\pi z\cot\pi z\\
&=\lim_{z\to 0}\frac{\pi^2\left(\pi z\cos\pi z-\sin\pi z\right)}{\sin^3\pi z}\\
&=\lim_{z\to 0}\frac{\pi^2\left(\pi z(1-\frac{\pi^2z^2}{2!}+\cdots)-(\pi z-\frac{\pi^3z^3}{3!}+\cdots)\right)}{\sin^3\pi z}\\
&=\lim_{z\to 0}\pi^2\cdot\frac{-\frac{\pi^3z^3}{3}+\text{higher-order terms}}{\sin^3\pi z}=-\frac{\pi^2}{3}.
\end{aligned}$$

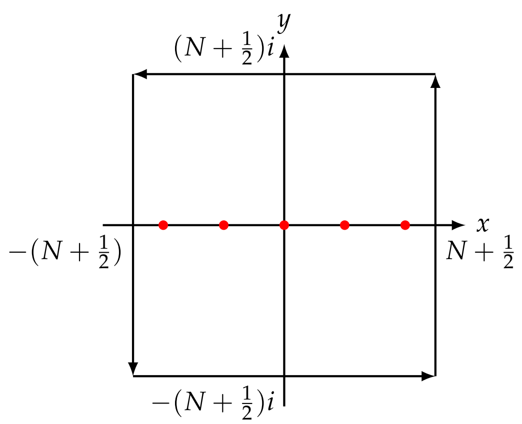
For any non-zero integer n, observe that:

$$\lim_{z\to n}(z-n)f(z)=\lim_{z\to n}\frac{\pi(z-n)\cos\pi z}{z^2\cdot\underbrace{(-1)^n\sin(\pi(z-n))}_{=\sin\pi z}}=\frac{(-1)^n}{n^2\cdot(-1)^n}=\frac{1}{n^2}.$$

Hence, n is a simple pole of f (for any $n\neq 0$), and $\operatorname{Res}(f,n)=\dfrac{1}{n^2}$.

Now consider the contour γ_N. The singularities it encloses are:

$$0,\pm 1,\pm 2\cdots,\pm N.$$



By Residue Theorem, we have:

$$\begin{aligned}\oint_{\gamma_N} f(z)\,dz &= 2\pi i \sum_{n=-N}^{N} \operatorname{Res}(f,n) \\ &= 2\pi i \left(\operatorname{Res}(f,0) + 2\left(1 + \frac{1}{2^2} + \cdots + \frac{1}{N^2}\right)\right) \\ &= 2\pi i \left(-\frac{\pi^2}{3} + 2\sum_{n=1}^{N} \frac{1}{n^2}\right).\end{aligned}$$

By rearrangement, we have the desired result:

$$\sum_{n=1}^{N} \frac{1}{n^2} = \frac{\pi^2}{6} + \frac{1}{2\pi i}\oint_{\gamma_N} f(z)\,dz.$$

The remaining task is to show:

$$\lim_{N\to\infty} \oint_{\gamma_N} \frac{\pi}{z^2}\cot \pi z\,dz = 0.$$

We do so by estimating the contour integral:

When $z \in \gamma_N$, we have $|z| \geq N + \frac{1}{2} > N$. It is also possible to show that $|\cot \pi z| < 2$ for any $z \in \gamma_N$ (this is left as an exercise). Therefore, on γ_N, we have the bound:

$$\left|\frac{\pi}{z^2}\cot \pi z\right| \leq \frac{2\pi}{N^2}.$$

The length of γ_N is $8N + 4$. By Lemma 3.6, we get:

$$\left|\oint_{\gamma_N} \frac{\pi}{z^2}\cot \pi z\,dz\right| \leq (8N+4)\cdot \frac{2\pi}{N^2} \to 0 \text{ as } N \to \infty,$$

completing the proof.

Exercise 4.23. Complete the detail of the above example that:

$$|\cot \pi z| < 2$$

for any $z \in \gamma_N$. [Hint: Write $z = x + yi$, and find an expression for $\cot \pi z$ in terms of x and y. Then, maximize $|\cot \pi z|$ on each side of the contour γ_N.]

Exercise 4.24. Use Residue Theorem to evaluate the following contour integrals:

(a) $\displaystyle\oint_{|z|=3} \frac{1}{z^2+1}\,dz$

(b) $\displaystyle\oint_{|z|=2} \frac{z^3+3z+1}{z^4-5z^2}\,dz$

(c) $\displaystyle\oint_{|z-i|=2} \frac{e^z+z}{(z-1)^4}\,dz$

(d) $\displaystyle\oint_{|z-i|=2} \frac{\sin z}{(z-i)^{4023}}\,dz$

(e) $\displaystyle\oint_{\gamma} \tan \pi z\,dz$ where γ is the rectangle contour with vertices:

$$(-2,0), \quad (2,0), \quad (2,1), \quad (-2,1).$$

Exercise 4.25. Let γ_N be the square contour with vertices $\pm(N+\frac{1}{2})\pi \pm (N+\frac{1}{2})\pi i$ where N is a positive integer. Show that:

$$\frac{1}{2\pi i}\oint_{\gamma_N}\frac{1}{z^2}\csc z\,dz = \frac{1}{6}+\frac{2}{\pi^2}\sum_{n=1}^{N}\frac{(-1)^n}{n^2}.$$

Hence, show that:

$$1-\frac{1}{2^2}+\frac{1}{3^2}-\frac{1}{4^2}+\cdots = \frac{\pi^2}{12}.$$

Exercise 4.26. Determine the residues of all isolated singularities of the function:

$$f(z) = \frac{1}{(2z-1)\sin \pi z}.$$

By considering a suitable contour integral of f, show that:

$$1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\cdots = \frac{\pi}{4}.$$

4.3.4. Evaluation of Real Integrals. Residues are often used to evaluate some difficult real integrals that physicists and engineers may encounter.

Example 4.12. Evaluate the real definite integral:

$$\int_0^{2\pi}\frac{1}{a-b\cos\theta}\,d\theta$$

where a and b are real numbers such that $0<b<a$.

Solution

The key trick is to express the real integral as a complex integral of the circle contour $|z|=1$, which is parametrized by $z=e^{i\theta}$ where $0\le\theta\le 2\pi$.

When $z=e^{i\theta}$ is on the contour $\{|z|=1\}$, we have

$$\cos\theta = \frac{e^{i\theta}+e^{-i\theta}}{2} = \frac{1}{2}\left(z+\frac{1}{z}\right)$$

$$dz = ie^{i\theta}\,d\theta \Longrightarrow d\theta = \frac{1}{ie^{i\theta}}\,dz = \frac{1}{iz}\,dz$$

Therefore, the real integral can be written as a complex integral as:

$$\int_0^{2\pi}\frac{1}{a-b\cos\theta}\,d\theta = \oint_{|z|=1}\frac{1}{a-\frac{b}{2}\left(z+\frac{1}{z}\right)}\cdot\frac{1}{iz}\,dz = 2i\oint_{|z|=1}\frac{1}{bz^2-2az+b}\,dz.$$

We can then use residue theory to evaluate the complex integral. The singularities of the integrand are roots of the quadratic equation $bz^2-2az+b=0$, which are:

$$\omega_1 = \frac{a-\sqrt{a^2-b^2}}{b} \quad\text{and}\quad \omega_2 = \frac{a+\sqrt{a^2-b^2}}{b}.$$

Note that $a>b$, so both roots are real. We further observe that:

$$\omega_2 > \frac{a+0}{b} > 1 \quad\text{and}\quad \omega_1\omega_2 = 1,$$

and so $|\omega_1|<1$. Therefore, ω_1 is the only singularity enclosed by the contour $|z|=1$. As ω_1 and ω_2 are distinct, they are simple poles, and so the contour integral is given by:

$$\begin{aligned}\oint_{|z|=1} \frac{1}{bz^2 - 2az + b}\,dz &= \oint_{|z|=1} \frac{1}{b(z-\omega_1)(z-\omega_2)}\,dz \\ &= \oint_{|z|=1} \frac{\frac{1}{b(z-\omega_2)}}{z-\omega_1}\,dz = 2\pi i \left[\frac{1}{b(z-\omega_2)}\right]_{z=\omega_1} \\ &= \frac{2\pi i}{b(\omega_1-\omega_2)} = -\frac{\pi i}{\sqrt{a^2-b^2}}.\end{aligned}$$

Hence, the real integral is given by:

$$\int_0^{2\pi} \frac{1}{a - b\cos\theta}\,d\theta = -2i \cdot \frac{\pi i}{\sqrt{a^2-b^2}} = \frac{2\pi}{\sqrt{a^2-b^2}}.$$

Before we proceed to the next example, let's first prove a useful observation which will come in handy later on.

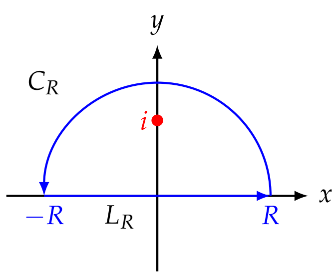
Exercise 4.27. Show that the function e^{iz} is bounded on the upper-half plane, i.e. there exists $M > 0$ such that $\left|e^{iz}\right| \leq M$ whenever $\mathrm{Im}(z) \geq 0$. On the other hand, show that the function $= \cos z$ is unbounded on the upper-half plane.

Example 4.13. Evaluate the following real integral:

$$\int_{-\infty}^{\infty} \frac{\cos x}{1+x^2}\,dx.$$

Solution

Let's consider the following semi-circle contour:



Denote C_R to be the (open) semi-circle with radius R, L_R to be the straight-path from $-R$ to R, and γ_R to be the closed semi-circular path $C_R + L_R$. We consider this contour because

$$\int_{-\infty}^{\infty} \frac{\cos x}{1+x^2}\,dx = \lim_{R\to+\infty} \int_{-R}^{R} \frac{\cos x}{1+x^2}\,dx = \lim_{R\to+\infty} \int_{L_R} \frac{\cos z}{1+z^2}\,dz.$$

Note that:

$$\oint_{\gamma_R} \frac{\cos z}{1+z^2}\,dz = \int_{L_R} \frac{\cos z}{1+z^2}\,dz + \int_{C_R} \frac{\cos z}{1+z^2}\,dz.$$

The γ_R-integral can be computed using residues. If we are able to show the C_R-integral tends to 0 as $R \to +\infty$, then one can determine our desired limit $\lim\limits_{R\to+\infty} \int_{L_R} \frac{\cos z}{1+z^2}\,dz$.

Unfortunately, it is not possible to bound $\frac{\cos z}{1+z^2}$ as $\cos z$ is unbounded according to Exercise 4.27. One trick to get around with this issue is to consider the following function instead:

$$f(z) = \frac{e^{iz}}{1+z^2}.$$

When $z \in L_R$, we have $z = x + 0i$ and so:

$$f(z) = \frac{e^{ix}}{1+x^2} = \frac{\cos x + i\sin x}{1+x^2} \implies \int_{L_R} f(z)\,dz = \int_{-R}^{R} \frac{\cos x + i\sin x}{1+x^2}\,dx.$$

If we are able to find out $\lim_{R\to+\infty} \int_{L_R} f(z)\,dz$, then one can recover the value of $\int_{-\infty}^{\infty} \frac{\cos x}{1+x^2}$ by simply taking the real-part of $\lim_{R\to+\infty} \int_{L_R} f(z)\,dz$.

By considering the contour $\gamma_R = C_R + L_R$, we have:

$$\oint_{\gamma_R} f(z)\,dz = \int_{L_R} f(z)\,dz + \int_{C_R} f(z)\,dz.$$

The only singularity enclosed by γ_R is i (when R is sufficiently large), so:

$$\oint_{\gamma_R} f(z)\,dz = 2\pi i \operatorname{Res}(f,i) = 2\pi i \cdot \frac{1}{2ie} = \frac{\pi}{e}.$$

Next we show the C_R-integral converges to 0 as $R \to \infty$. From Exercise 4.27, the term e^{iz} is bounded on the upper-half plane, and so whenever $z \in C_R$, we have:

$$\left|\frac{e^{iz}}{1+z^2}\right| \le \frac{M}{|1+z^2|} \le \frac{M}{\left||z|^2-1\right|} = \frac{M}{R^2-1},$$

where M is an upper bound of $\left|e^{iz}\right|$ on the upper-half plane. Therefore, by Lemma 3.6, we get the estimate:

$$\left|\int_{C_R} \frac{e^{iz}}{1+z^2}\,dz\right| \le \pi R \cdot \frac{M}{R^2-1} \to 0 \quad \text{as } R \to +\infty.$$

Therefore, we get:

$$\begin{aligned} \lim_{R\to\infty} \int_{L_R} f(z)\,dz &= \lim_{R\to\infty} \oint_{\gamma_R} f(z)\,dz - \lim_{R\to\infty} \int_{C_R} f(z)\,dz \\ \int_{-\infty}^{\infty} \frac{\cos x + i\sin x}{1+x^2}\,dx &= \frac{\pi}{e} - 0 = \frac{\pi}{e}. \end{aligned}$$

This shows:

$$\int_{-\infty}^{\infty} \frac{\cos x}{1+x^2} = \frac{\pi}{e}.$$

Before we give another example, we recall some fundamental facts that:

- For any $z \neq 0$, the principal argument $\operatorname{Arg}(z)$ is in $(-\pi, \pi]$.
- $\operatorname{Log}(z) = \ln|z| + i\operatorname{Arg}(z)$ for any $z \neq 0$
- $\operatorname{Log}(z)$ is holomorphic on $\mathbb{C}\backslash(-\infty, 0]$.

Therefore, if we apply Cauchy's integral formula or residue theory for an integral involving $\operatorname{Log}(z)$, then we need to make sure the closed curve γ lies in $\mathbb{C}\backslash(-\infty, 0]$. As such, we cannot apply residue methods with a semi-circle contour as in the previous example. Nonetheless, this kind of semi-circle contour is very useful when dealing with real integrals over $(-\infty, \infty)$.

To get around with this issue, we can define a different branch of logarithm by the following. For any $z \neq 0$, we let

$$\operatorname{Log}_{-\pi/2}(z) := \ln|z| + i\theta(z) \tag{4.5}$$

where $\theta(z)$ is the unique angle in $[-\frac{\pi}{2}, \frac{3\pi}{2})$ such that $z = |z|\, e^{i\theta(z)}$. By doing so, we still have $e^{\mathrm{Log}_{-\pi/2}(z)} = z$. The notable difference from $\mathrm{Log}(z)$ is that now $\mathrm{Log}_{-\pi/2}(z)$ is holomorphic on $\mathbb{C}\backslash\{0 + yi : y \leq 0\}$, the yellow region below.

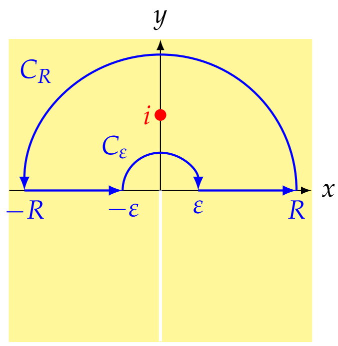


Figure 4.1. Domain of $\mathrm{Log}_{-\pi/2}(z)$

Exercise 4.28. Determine the value of $\mathrm{Log}_{-\pi/2}(z)$ when:

(a) $z = i$

(b) $z = x + 0i$ where $x > 0$

(c) $z = x + 0i$ where $x < 0$

Example 4.14. Let α be a real constant in $(0, 1)$. Evaluate the real integral:

$$I := \int_0^\infty \frac{1}{x^\alpha(1 + x^2)}\, dx.$$

Solution

First observe that for any $x > 0$, we have:

$$x^\alpha = e^{\alpha \ln x} = e^{\alpha \mathrm{Log}_{-\pi/2}(x)}$$

where $\mathrm{Log}_{-\pi/2}$ is the special branch of logarithm defined in (4.5). It prompts us to consider a contour integral of the function:

$$f(z) = \frac{1}{e^{\alpha \mathrm{Log}_{-\pi/2}(z)}(1 + z^2)}.$$

We pick a contour as shown in Figure 4.1, where C_R and C_ε are semi-circles with radii R and ε respectively. Since the closed contour $\gamma_{R,\varepsilon} := [-R, -\varepsilon] + C_\varepsilon + [\varepsilon, R] + C_R$ lies completely inside the domain of $\mathrm{Log}_{-\pi/2}(z)$, by Residue Theorem, we have:

$$\begin{aligned}\oint_{\gamma_{R,\varepsilon}} \frac{1}{e^{\alpha \mathrm{Log}_{-\pi/2}(z)}(1 + z^2)}\, dz &= 2\pi i \,\mathrm{Res}(f, i) = 2\pi i \cdot \left.\frac{1}{e^{\alpha \mathrm{Log}_{-\pi/2}(z)}(z + i)}\right|_{z=i} \\ &= \frac{2\pi i}{2ie^{\alpha(\ln|i| + \frac{\pi}{2}i)}} = \frac{\pi}{e^{\frac{\alpha\pi}{2}i}}.\end{aligned}$$

On the other hand, the $\gamma_{R,\varepsilon}$-integral can break down into:

(4.6)
$$\oint_{\gamma_{R,\varepsilon}} \frac{1}{e^{\alpha \mathrm{Log}_{-\pi/2}(z)}(1 + z^2)}\, dz = \left(\int_{-R}^{-\varepsilon} + \int_{C_\varepsilon} + \int_\varepsilon^R + \int_{C_R}\right) \frac{1}{e^{\alpha \mathrm{Log}_{-\pi/2}(z)}(1 + z^2)}\, dz$$

When $z = x + 0i \in [\varepsilon, R]$, the integrand is simply:

$$\frac{1}{e^{\alpha \mathrm{Log}_{-\pi/2}(x)}(1+x^2)} = \frac{1}{x^\alpha(1+x^2)}.$$

Hence,

$$\lim_{\varepsilon\to 0}\lim_{R\to\infty}\int_\varepsilon^R \frac{1}{e^{\alpha \mathrm{Log}_{-\pi/2}(z)}(1+z^2)}\,dz = \int_0^\infty \frac{1}{x^\alpha(1+x^2)}\,dx =: I.$$

When $z = x + 0i \in [-R, -\varepsilon]$, the integrand becomes:

$$\frac{1}{e^{\alpha \mathrm{Log}_{-\pi/2}(x)}(1+x^2)} = \frac{1}{e^{\alpha(\ln|x|+\pi i)}(1+x^2)} = \frac{1}{e^{\alpha\pi i}}\cdot\frac{1}{|x|^\alpha(1+x^2)}.$$

Hence,

$$\lim_{\varepsilon\to 0}\lim_{R\to\infty}\int_{-R}^{-\varepsilon} \frac{1}{e^{\alpha \mathrm{Log}_{-\pi/2}(z)}(1+z^2)}\,dz = \frac{1}{e^{\alpha\pi i}}\int_{-\infty}^0 \underbrace{\frac{1}{|x|^\alpha(1+x^2)}}_{\text{even function}}\,dx = \frac{I}{e^{\alpha\pi i}}.$$

We are left to analyze the two semi-circular integrals. We will show that they tend to 0 as $\varepsilon \to 0$ and $R \to \infty$.

When $z \in C_\varepsilon$, we have:

$$\left|\frac{1}{e^{\alpha \mathrm{Log}_{-\pi/2}(z)}(1+z^2)}\right| \leq \left|e^{-\alpha(\ln|z|+i\theta(z))}\right|\cdot = \frac{1}{\left|1-|z|^2\right|} = \frac{e^{-\alpha\ln\varepsilon}}{1-\varepsilon^2} = \frac{\varepsilon^{-\alpha}}{1-\varepsilon^2}.$$

By Lemma 3.6, we get:

$$\left|\int_{C_\varepsilon} \frac{1}{e^{\alpha \mathrm{Log}_{-\pi/2}(z)}(1+z^2)}\,dz\right| \leq \frac{\varepsilon^{-\alpha}}{1-\varepsilon^2}\cdot\pi\varepsilon = \frac{\pi\varepsilon^{1-\alpha}}{1-\varepsilon^2} \to 0 \quad \text{as } \varepsilon \to 0.$$

Similarly when $z \in C_R$, we have:

$$\left|\frac{1}{e^{\alpha \mathrm{Log}_{-\pi/2}(z)}(1+z^2)}\right| \leq \left|e^{-\alpha(\ln|z|+i\theta(z))}\right|\cdot\left|\frac{1}{\left|1-|z|^2\right|}\right| = \frac{R^{-\alpha}}{R^2-1}.$$

By Lemma 3.6, we have the estimate:

$$\left|\int_{C_\varepsilon} \frac{1}{e^{\alpha \mathrm{Log}_{-\pi/2}(z)}(1+z^2)}\,dz\right| \leq \frac{R^{-\alpha}}{R^2-1}\cdot\pi R = \frac{\pi R^{1-\alpha}}{R^2-1} \to 0 \quad \text{as } R \to \infty.$$

Finally, by letting $\varepsilon \to 0$ and $R \to \infty$ on both sides of (4.6), we get:

$$\frac{\pi}{e^{\frac{\alpha\pi}{2}i}} = I + \frac{I}{e^{\alpha\pi i}}.$$

Solving for I, we get:

$$I = \frac{\pi}{e^{\frac{\alpha\pi}{2}i}\left(1+e^{-\alpha\pi i}\right)} = \frac{\pi}{e^{\frac{\alpha\pi}{2}i}+e^{-\frac{\alpha\pi}{2}i}} = \frac{\pi}{2\cos\frac{\alpha\pi}{2}} = \frac{\pi}{2}\sec\frac{\alpha\pi}{2}.$$

Exercise 4.29. Evaluate the following real integrals using residue methods:

(a) $\displaystyle\int_0^{2\pi} \frac{1}{(a+b\cos\theta)^2}\,d\theta$ where $a > b > 0$.

(b) $\displaystyle\int_0^{2\pi} \frac{1}{1-2a\cos\theta+a^2}\,d\theta$ where $a \in \mathbb{R}$ and $a \neq \pm 1$.

(c) $\displaystyle\int_0^{\infty} \frac{x^2}{(x^2+a^2)^2}\,dx$ where $a > 0$.

(d) $\displaystyle\int_0^{\infty} \frac{1}{(x^2+1)^n}\,dx$ where $n \in \mathbb{N}$.

(e) $\displaystyle\int_0^{\infty} \frac{\cos ax}{x^2+b^2}\,dx$ where a and b are positive real numbers

(f) $\displaystyle\int_0^{\infty} \frac{\sin ax}{x(x^2+1)}\,dx$ where a is a positive real number.

(g) $\displaystyle\int_0^{\infty} \frac{\ln x}{x^2+a^2}\,dx$ where $a > 0$.

(h) $\displaystyle\int_0^{\infty} \frac{1}{x^\alpha(1+x^4)}\,dx$ where $\alpha \in (0,1)$.

Exercise 4.30. Show that for any $t \in \mathbb{R}$, we have:

$$\int_{-\infty}^{\infty} \frac{e^{itx}}{x^2+1}\,dx = \pi e^{-|t|}.$$

Exercise 4.31. Show that:

$$\int_{-\infty}^{\infty} \frac{1}{(1+x^2)^{n+1}}\,dx = \frac{(2n-1)!!}{(2n)!!}\pi.$$