Appendix A

Results from MATH 3033/3043

In this appendix we list some important concepts and theorems from MATH 3033/3043 that we will use frequently in this course. Proofs are all omitted since they are essentially the same as in the real case. This appendix is intended to be brief (no worked example here). For detail, please consult Chapter 10 of MATH 3033, or Chapter 4 in MATH 3043.

Definition A.1 (Uniform Convergence)**.** A sequence of functions $f_n(z)$ is said to converge to $f(z)$ *uniformly on* Ω if

$$\sup\{|f_n(z) - f(z)| : z \in \Omega\} \to 0 \quad \text{as } n \to \infty.$$

A (pointwise) convergent series $\sum_{n=1}^{\infty} f_n(z)$ is said to *converge uniformly on* Ω if the N-th partial sum $\sum_{n=1}^{N} f_n(z)$ converges uniformly on Ω as $N \to \infty$. In other words:

$$\sup\left\{\left|\sum_{n=1}^{N} f_n(z) - \sum_{n=1}^{\infty} f_n(z)\right| : z \in \Omega\right\} \to 0 \quad \text{as } N \to \infty.$$

It is sometimes difficult to show a series converges uniformly from the definition. Fortunately, we have the following useful test:

Theorem A.2 (Weierstrass' M-test)**.** *Consider a series* $\sum_{n=1}^{\infty} f_n(z)$ *defined on* Ω*. If there exists a sequence of real numbers* $M_n \in \mathbb{R}$*, independent of* z*, such that:*

- $|f_n(z)| \leq M_n$ *for any* $z \in \Omega$ *and any* n*, and*
- *the series* $\sum_{n=1}^{\infty} M_n$ *converges,*

then $\sum_{n=1}^{\infty} f_n(z)$ *converges uniformly on* Ω*.*

There are many nice consequences if a series or sequence converges uniformly, namely we can switch the integral, limit and summation signs quite freely:

Proposition A.3. *Suppose $f_n(z)$ converges uniformly on Ω to the limit function $f(z)$, then:*

- *If f_n are continuous on Ω for all n, then f is also continuous on Ω.*
- *For any $\alpha \in \Omega$, we have*
$$\lim_{z\to\alpha}\lim_{n\to\infty} f_n(z) = \lim_{n\to\infty}\lim_{z\to\alpha} f_n(z).$$
- *Let $[a,b]$ be a bounded interval in $\mathbb{R}$, and $f_n(t)$'s be integrable functions on $[a,b]$ If $f_n(t)$ converges uniformly to $f(t)$ on $[a,b]$, then*
$$\lim_{n\to\infty}\int_a^b f_n(t)\,dt = \int_a^b \lim_{n\to\infty} f_n(t)\,dt.$$
- *Let γ be a curve in $\mathbb{C}$ of finite length, and $f_n(z)$'s be integrable functions on γ. If $f_n(z)$ converges uniformly to $f(z)$ on Ω, then*
$$\lim_{n\to\infty}\int_\gamma f_n(z)\,dz = \int_\gamma \lim_{n\to\infty} f_n(z)\,dz.$$

Analogous results hold for uniform convergence series. For instance, if $\sum_{n=1}^{\infty} f_n(z)$ converges uniformly on Ω, then for any curve γ in Ω of finite length, we have:
$$\int_\gamma \sum_{n=1}^{\infty} f_n(z)\,dz = \sum_{n=1}^{\infty}\int_\gamma f_n(z)\,dz.$$

In the above proposition, the conditions that $[a,b]$ is a finite interval, and γ is a curve of finite length are necessary. While we mostly encounter curves of finite lengths for contour integrals, we will occasionally come across real intervals of *unbounded intervals*. In such case, uniform convergence is not sufficient to guarantee switching of the integral and summation signs! Fortunately, there is another tool to deal with improper integrals, namely Lebesgue Dominated Convergence Theorem (LDCT), which stems from measure theory:

Theorem A.4 (Lebesgue Dominated Convergence Theorem)**.** *Let $f_n(t) : (a,b) \to \mathbb{C}$ be a sequence of measurable functions (including continuous functions) defined on a possibly infinite interval $(a,b) \subset \mathbb{R}$. Suppose:*

- *$f_n(t) \to f(t)$ pointwise on every $t \in (a,b)$, and*
- *there exists an integrable function $h : (a,b) \to \mathbb{R}$ independent of n such that*
$$|f_n(t)| \le h(t) \quad \text{for any } t \in (a,b) \text{ and any } n,$$

then we have
$$\lim_{n\to\infty}\int_a^b f_n(t)\,dt = \int_a^b \lim_{n\to\infty} f_n(t)\,dt.$$

Consider a series $\sum_{n=1}^{\infty} g_n(t)$ where $g_n : (a,b) \to \mathbb{C}$ are measurable. Suppose
$$\sum_{n=1}^{\infty}\int_a^b |g_n(t)|\,dt < \infty,$$
then we have:
$$\int_a^b \sum_{n=1}^{\infty} g_n(t)\,dt = \sum_{n=1}^{\infty}\int_a^b g_n(t)\,dt.$$

Recall from MATH 3033/3043 that even if $f_n(x)$ converges uniformly on (a,b) to $f(x)$, the derivatives $f_n'(x)$ may not converge to f'. Sometimes, the limit of f_n' may not even be differentiable. Likewise, even when the sum $\sum_{n=1}^{\infty} f_n(x)$ converges uniformly on (a,b), term-by-term differentiation

$$\frac{d}{dx}\sum_{n=1}^{\infty} f_n(x) = \sum_{n=1}^{\infty} f_n'(x)$$

may not hold. We will not pursue a full discussion about term-by-term differentiation here, but we would like to remind you one fact that you can always do term-by-term differentiation for a convergent *power series*.

Proposition A.5. *Suppose the power series* $f(z) = \sum_{n=0}^{\infty} c_n(z-z_0)^n$ *converges on* $B_r(z_0)$*, then we have:*

$$f'(z) = \sum_{n=0}^{\infty} \frac{d}{dz} c_n(z-z_0)^n = \sum_{n=1}^{\infty} nc_n(z-z_0)^{n-1}$$

at every $z \in B_r(z_0)$*.*

When using Morera's Theorem, we often consider a double integral of the form:

$$\oint_T \int_a^b f(z,t)\, dtdz.$$

It we can switch the two integral signs, and it happens that $f(z,t)$ is a holomorphic function for each fixed $t \in [a,b]$, then we have:

$$\oint_T \int_a^b f(z,t)\, dtdz = \int_a^b \oint_T f(z,t)\, dzdt = \int_a^b 0\, dt = 0.$$

The question is whether we can switch the two integral signs. It thanks for the following (special case) of Fubini's Theorem

Theorem A.6 (Fubini's Theorem: special case)**.** *Suppose* $f(z,t) : \Omega \times I \to \mathbb{C}$ *is a continuous function, where* $\Omega \subset \mathbb{C}$ *and* I *is an interval (possibly infinite) in* $\mathbb{R}$*. Let* γ *be a curve in* Ω*. If one of the following is finite:*

$$\int_\gamma \int_I |f(z,t)|\, dt\, |dz| \quad \text{or} \quad \int_I \int_\gamma |f(z,t)|\, |dz|\, dt.$$

Then, we have:

$$\int_\gamma \int_I f(z,t)\, dtdz = \int_I \int_\gamma f(z,t)\, dzdt.$$

Here $|dz|$ *means* $\sqrt{(dx)^2 + (dy)^2}$*.*