Chapter 3

# Contour Integrals

We start discussing complex integrations in this chapter. Given a function $f : \Omega \subset \mathbb{C} \to \mathbb{C}$ and a $C^1$ curve $\gamma$ in the domain of $f$, the *contour integral* of $f$ over $\gamma$ is denoted by:

$$\int_\gamma f(z)\,dz.$$

We will learn how they are defined and how they can be computed soon. In the first glance, it appears quite similar to line integrals in Multivariable Calculus. However, when combining with properties of holomorphic functions, there are many beautiful and amazing results concerning complex contour integrals which did not appear in line integrals. One notable result is Cauchy's integral formula, an elegant theorem which leads to many important results in Complex Analysis and beyond.

## 3.1. Complex Integrations

### 3.1.1. Contour Integrals.

Consider a $C^1$ curve $\gamma$ in $\mathbb{C}$ parametrized by:

$$z(t) = x(t) + iy(t), \quad t \in [a,b].$$

The differential $dz$ is regarded as:

$$dz = \frac{dz}{dt}\,dt = \left(x'(t) + iy'(t)\right)\,dt.$$

For example, if $\gamma$ is the unit circle centered at the origin, then it is parametrized by:

$$z(t) = \cos t + i \sin t = e^{it}, \quad t \in [0, 2\pi].$$

Hence, we have $dz = \dfrac{d(e^{it})}{dt}\,dt = ie^{it}\,dt$.

**Definition 3.1** (Contour Integrals)**.** Let $f : \Omega \to \mathbb{C}$ be a continuous function on the open domain $\Omega \subset \mathbb{C}$, and $\gamma$ be a $C^1$ curve in $\Omega$. Suppose $\gamma$ is parametrized by

$$z(t) = x(t) + iy(t), \quad t \in [a,b],$$

then the *contour integral of $f$ over $\gamma$* is denoted and defined by:

$$\int_\gamma f(z)\,dz := \int_a^b f(z(t))\,\underbrace{z'(t)\,dt}_{dz}.$$

**Remark 3.2.** If $\gamma$ is a piecewise $C^1$ curve, meaning that it can be decomposed into $\gamma = \gamma_1 + \ldots + \gamma_k$ where each of $\gamma_1, \ldots, \gamma_k$ is $C^1$, and that the whole curve $\gamma$ is continuous, then we define:

$$\int_\gamma f(z)\,dz = \int_{\gamma_1} f(z)\,dz + \ldots + \int_{\gamma_k} f(z)\,dz.$$

Furthermore, if $\gamma$ is closed, we usually denote the contour integral by:

$$\oint_\gamma f(z)\,dz.$$

**Example 3.1.** Compute the line integral $\oint_\gamma f(z)\,dz$ for each of the functions below. Here $\gamma$ is the circle with radius 2 centered at the origin.

(a) $f(z) = z^2$

(b) $f(z) = \dfrac{1}{z}$

(c) $f(z) = \overline{z}$

**Solution**

$\gamma$ can be parametrized by:

$$z(t) = 2e^{it}, t \in [0, 2\pi].$$

Therefore $dz = 2ie^{it}\,dt$.

(a)

$$\begin{aligned}\oint_\gamma z^2\,dz &= \int_0^{2\pi} \underbrace{\left(2e^{it}\right)^2}_{z^2} \cdot \underbrace{2ie^{it}\,dt}_{dz} = \int_0^{2\pi} 8ie^{3it}\,dt \\ &= 8i\left[\frac{1}{3i}e^{3it}\right]_{t=0}^{t=2\pi} \\ &= \frac{8}{3}\left(e^{6\pi i} - e^0\right) = \frac{8}{3}(1-1) = 0.\end{aligned}$$

(b)

$$\oint_\gamma \frac{1}{z}\,dz = \int_0^{2\pi} \frac{1}{2e^{it}} \cdot 2ie^{it}\,dt = \int_{t=0}^{t=2\pi} i\,dt = 2\pi i.$$

(c)

$$\oint_\gamma \overline{z}\,dz = \int_0^{2\pi} 2e^{-it} \cdot 2e^{it}\,dt = \int_0^{2\pi} 4\,dt = 8\pi i.$$

**Remark 3.3.** In part (a) of the above example, we have used the fact that $\frac{d}{dt}\left(\frac{1}{3i}e^{3it}\right) = e^{3it}$, and also Fundamental Theorem of Calculus. In general, just like in the real case, if $F(t)$ is a differentiable function of $t$ on $[a, b]$ such that $F'(t) = \varphi(t)$ on $[a, b]$, then we have

$$\int_{t=a}^{t=b} \varphi(t)\,dt = F(b) - F(a).$$

However, we sometimes need to be more careful when applying this. Try to find out what's wrong with the calculation below:

$$\oint_{|z|=1} \frac{1}{1-2z}\,dz = \int_0^{2\pi} \frac{1}{1-2e^{it}} ie^{it}\,dt = \left[-\frac{1}{2i}\mathrm{Log}(1-2e^{it})\right]_0^{2\pi}$$
$$= -\frac{1}{2i}\left(\mathrm{Log}(-1) - \mathrm{Log}(-1)\right) = 0???$$

**Example 3.2.** Consider the line segment $L$ from a point $z_1$ to a point $z_2$ in $\mathbb{C}$. Compute the following contour integral (in terms of $z_1$ and $z_2$):

$$\int_L e^z\,dz.$$

**Solution**

First we parametrize $L$:

$$z(t) = (1-t)z_1 + tz_2, \quad t \in [0,1].$$

Then, we have $dz = (z_2 - z_1)\,dt$, and so:

$$\int_L e^z\,dz = \int_0^1 e^{z_1+t(z_2-z_1)} \cdot (z_2 - z_1)\,dt$$
$$= \left[\frac{1}{z_2 - z_1} e^{z_1+t(z_2-z_1)} \cdot (z_2 - z_1)\right]_0^1$$
$$= e^{z_2} - e^{z_1}.$$

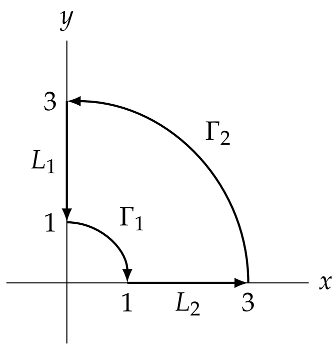



**Figure 3.1.** the path in Example 3.1

**Exercise 3.1.** Compute the contour integrals

$$\oint_\gamma \frac{1}{z^2}\,dz, \quad \oint_\gamma \bar{z}\,dz \quad \text{and} \quad \oint_\gamma |z|\,dz$$

where $\gamma = \Gamma_1 + L_2 + \Gamma_2 + L_1$ is the curve in Figure 3.1.

**3.1.2. Primitive Functions.** In Calculus I, we learned that if $F'(x) = f(x)$ on $x \in [a,b]$, then:

$$\int_a^b f(x)\,dx = F(b) - F(a).$$

This is the celebrated Fundamental Theorem of Calculus. In Complex Analysis, we have an analogous result:

**Theorem 3.4.** *Let $f : \Omega \to \mathbb{C}$ be a continuous function defined on an open domain $\Omega \subset \mathbb{C}$, and $\gamma$ be a piecewise $C^1$ curve in $\Omega$ with starting point $z_1$ and ending point $z_2$. If $F : \Omega \to \mathbb{C}$ is a (single-valued) holomorphic function on $\Omega$ such that $F'(z) = f(z)$ for every $z \in \Omega$, then we have:*

$$\int_\gamma f(z)\,dz = F(z_2) - F(z_1).$$

**Proof.** First assume that $\gamma$ is $C^1$. Suppose the path $\gamma$ can be parametrized by:

$$z(t) = x(t) + iy(t), \quad t \in [a, b].$$

Then, we have $dz = z'(t)\,dt$, and hence:

$$\begin{aligned}
\int_\gamma f(z)\,dz &= \int_a^b f(z(t)) \cdot z'(t)\,dt \\
&= \int_a^b \underbrace{F'(z(t))}_{f(z(t))} \cdot z'(t)\,dt \\
&= \int_a^b \frac{d}{dt} F(z(t))\,dt && \text{(chain rule)} \\
&= F(z(b)) - F(z(a)) \\
&= F(z_2) - F(z_1).
\end{aligned}$$

If $\gamma$ is only piecewise $C^1$, we can decompose $\gamma = \gamma_1 + \cdots + \gamma_k$ so that each $\gamma_i$ is $C^1$. Then, one can argue as above for each $\gamma_i$, and finally obtain the desired result by adding a telescope sum. □

**Remark 3.5.** If such an $F(z)$ in Theorem 3.4 exists, then we call $F(z)$ a *primitive function* of $f(z)$.

The above theorem is particularly useful when the anti-derivative of $f$ is easy to find. For example, if $\gamma$ is any continuous piecewise $C^1$ path from $z_1$ to $z_2$, we can find easily that:

$$\int_\gamma z^2\,dz = \left[\frac{z^3}{3}\right]_{z_1}^{z_2} = \frac{z_2^3 - z_1^3}{3}$$

$$\int_\gamma e^z\,dz = [e^z]_{z_1}^{z_2} = e^{z_2} - e^{z_1}.$$

In particular, if $C$ is a closed path, then we have:

$$\oint_C z^2\,dz = 0 \quad \text{and} \quad \oint_C e^z\,dz = 0.$$

**Exercise 3.2.** Let $\gamma_1$ be the path which starts from $(0, 0)$, first to $(1, 1)$, then to $(0, 2)$. Let $\gamma_2$ be the path which starts from $(0, 0)$, then straight to $(0, 2)$. Verify the following by direct computations:

$$\int_{\gamma_1} \cos\frac{\pi z}{2}\,dz = \int_{\gamma_2} \cos\frac{\pi z}{2}\,dz.$$

Then, verify that Theorem 3.4 gives the same result.

However, it is important to note that Theorem 3.4 requires the curve $\gamma$ to be inside $\Omega$ (on which $F'(z) = f(z)$ holds). Let's consider the function $f(z) = \frac{1}{z}$. Although we usually simply write $\frac{d}{dz}\text{Log}(z) = \frac{1}{z}$, it is only true for $z \in \mathbb{C}\backslash\{x + 0i : x \le 0\}$ since $\text{Log}(z)$ is not continuous on the negative $x$-axis.

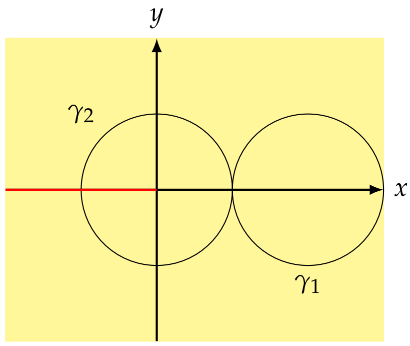
Therefore, we can only apply Theorem 3.4 when the curve $\gamma$ lies inside $\Omega := \mathbb{C}\backslash\{x + 0i : x \le 0\}$. For instance, we still have

$$\oint_{\gamma_1} \frac{1}{z} = 0$$

where $\gamma_1$ is the unit circle centered at $2 + 0i$ with radius 1. This closed curve $\gamma_1$ is contained inside $\Omega$.

However, it is incorrect to claim $\oint_{\gamma_2} \frac{1}{z} = 0$ where $\gamma_2$ is the unit circle centered at the origin. The reason is that this closed curve passes through the negative $x$-axis (hence not contained inside $\Omega$). In fact we can directly verify that:

$$\oint_{\gamma_2} \frac{1}{z} = 2\pi i.$$




Fortunately, we can still apply Theorem 3.4 on $f(z) = \frac{1}{z^2}$ when the integration curve $\gamma$ does not pass through the origin. The reason is that $F(z) = -\frac{1}{z}$ is a primitive function for $f$ such that $F'(z) = f(z)$ holds on $\mathbb{C}\backslash\{0\}$. Therefore, we have:

$$\oint_{\gamma} \frac{1}{z^2} = 0$$

for any closed curve $\gamma$ not passing through the origin. Also, for a path $L$ in $\mathbb{C}\backslash\{0\}$ connecting $z_1$ to $z_2$, we have:

$$\int_L \frac{1}{z^2}\, dz = \left[-\frac{1}{z}\right]_{z_1}^{z_2} = \frac{1}{z_1} - \frac{1}{z_2}.$$

**Exercise 3.3.** Consider the path $\gamma$ parametrized by:

$$z(t) = \cos^{3033} t + i \sin^{2033} t, \quad \text{where } t \in [0, \pi].$$

Find the contour integrals $\int_\gamma \frac{1}{z^{1014}}\, dz$ and $\int_\gamma (1 + iz)^{1013}\, dz$.

**Exercise 3.4.** Evaluate the integral $\int_\gamma |z|\, dz$ where $\gamma$ is each of the following:

(a) a line segment joining $-i$ to $i$.

(b) a counter-clockwise semi-circular path joining $-i$ to $i$

Does it exist an entire function $F : \mathbb{C} \to \mathbb{C}$ such that $F'(z) = |z|$ for any $z \in \mathbb{C}$? Why or why not?

**Exercise 3.5.** First verify that on an appropriate domain, we have:

$$\frac{d}{dz} i\left(\operatorname{Log}(i+z) - \operatorname{Log}(i-z)\right) = \frac{1}{1+z^2}.$$

Using this, show that:

$$\oint_{|z|=r} \frac{1}{1+z^2}\,dz = 0 \text{ when } r < 1.$$

In your solution, explain clearly where the condition $r < 1$ is needed.

**3.1.3. Integral Estimates.** Estimation of a contour integral is an important technique in Complex Analysis. It will appear in many parts of the course. If we know an upper bound for $|f(z)|$ on the curve $\gamma$, and the upper bound for the length of $\gamma$, then we are able to bound the contour integral $\int_\gamma f(z)\,dz$ without calculating it.

**Lemma 3.6.** *Let $f : \Omega \to \mathbb{C}$ be defined on an open domain $\Omega$. Suppse $\gamma$ is a curve in $\Omega$ such that:*

- *$|f(z)| \leq M$ for any $z \in \gamma$, and*
- *the arc-length of $\gamma$ is bounded above by $L$.*

*Then, we have:*

$$\left|\int_\gamma f(z)\,dz\right| \leq ML.$$

**Proof.** There is a nice trick in the proof that readers are recommended to learn. Let

$$I = \int_\gamma f(z)\,dz.$$

Express $I$ in polar form: $I = |I|\,e^{i\theta}$, then we have $e^{-i\theta} I = |I|$ which is real! Suppose $\gamma$ is parametrized by $z(t) = x(t) + iy(t)$ where $a \leq t \leq b$, then:

$$\begin{aligned}
e^{-i\theta} I = e^{-i\theta}\int_\gamma f(z)\,dz &= \int_\gamma e^{-i\theta} f(z)\,dz \\
&= \int_a^b \left[\operatorname{Re}\left(e^{-i\theta} f(z)\right) + i\operatorname{Im}\left(e^{-i\theta} f(z)\right)\right]\left(x'(t) + iy'(t)\right) dt \\
&= \int_a^b \left[\operatorname{Re}\left(e^{-i\theta} f(z)\right) x'(t) - \operatorname{Im}\left(e^{-i\theta} f(z)\right) y'(t)\right] dt.
\end{aligned}$$

The last equality above follows from the fact that $e^{-i\theta} I$ is real.

Then, we use Cauchy-Schwarz's inequality to bound the integrand:

$$\begin{aligned}
&\left|\operatorname{Re}\left(e^{-i\theta} f(z)\right) x'(t) - \operatorname{Im}\left(e^{-i\theta} f(z)\right) y'(t)\right| \\
&\leq \sqrt{\left(\operatorname{Re}\left(e^{-i\theta} f(z)\right)\right)^2 + \left(\operatorname{Im}\left(e^{-i\theta} f(z)\right)\right)^2}\sqrt{(x'(t))^2 + (y'(t))^2} \\
&= \left|e^{-i\theta} f(z)\right| |z'(t)| = |f(z)|\,|z'(t)| \leq M\,|z'(t)|\,.
\end{aligned}$$

Finally, we get:

$$\left|e^{-i\theta} I\right| \leq \int_a^b M\,|z'(t)| = ML,$$

and hence $|I| \leq ML$, completing the proof. □

**Remark 3.7.** If we estimate the integral $\left|\int_\gamma f(z)\,dz\right|$ in a more direct way by writing $f = u + iv$ and then consider the following:

$$\left|\int_\gamma f(z)\,dz\right| = \left|\int_\gamma (u+iv)(dx+idy)\right| = \left|\int_a^b (ux' - vy') + i(vx' + uy')\,dt\right|$$
$$= \sqrt{\left(\int_a^b (ux' - vy')\,dt\right)^2 + \left(\int_a^b (vx' + uy')\,dt\right)^2}.$$

then after applying Cauchy-Schwarz's inequality to each integral, the best we can achieve is

$$\left|\int_\gamma f(z)\,dz\right| \le \sqrt{2}ML,$$

which is weaker than the result in Lemma 3.6.

**Example 3.3.** Find an upper bound for the contour integral:

$$\left|\oint_{|z|=1} e^{\frac{1}{z}}\,dz\right|.$$

**Solution**

For any $z \in \mathbb{C}$ such that $|z| = 1$, we have:

$$e^{\frac{1}{z}} = e^{\frac{x}{x^2+y^2} - i\frac{y}{x^2+y^2}} = e^{x-iy} = e^x e^{-iy},$$
$$\left|e^{\frac{1}{z}}\right| = e^x \le e^1 = e.$$

Here we have used the fact that $-1 \le x \le 1$ along the curve $|z| = 1$.

Therefore, by Lemma 3.6, we have:

$$\left|\oint_{|z|=1} e^{\frac{1}{z}}\,dz\right| \le \underbrace{2\pi}_{L}\underbrace{e}_{M}.$$

**Example 3.4.** Show that:

$$\lim_{R\to+\infty} \oint_{|z|=R} \frac{1}{(z-1)^2}\,dz = 0.$$

**Solution**

We are interested in the limit when $R \to +\infty$, so we can assume $R > 1$ so that the contour circle $|z| = R$ does not pass through 1 (at which the integrand is undefined).

On the contour $|z| = R$, we have $|z-1| \ge R-1$ (draw a diagram to convince yourself on that), so we have:

$$\left|\frac{1}{(z-1)^2}\right| = \frac{1}{|z-1|^2} \le \underbrace{\frac{1}{(R-1)^2}}_{M} \quad \text{on } |z| = R.$$

The length of the contour $|z| = R$ is $2\pi R$. Hence, by Lemma 3.6, we get

$$\left|\oint_{|z|=R} \frac{1}{(z-1)^2}\,dz\right| \le 2\pi R \cdot \frac{1}{(R-1)^2}.$$

From elementary calculus, we have $\lim_{R\to+\infty} \frac{2\pi R}{(R-1)^2} = 0$, and the desired result follows from the squeeze theorem.

**Exercise 3.6.** Let $f : \mathbb{C} \to \mathbb{C}$ be a continuous function, and consider a fixed point $\alpha \in \mathbb{C}$. Show that:

$$\left| \oint_{|z|=R} \frac{f(z)}{z-\alpha}\, dz \right| \leq \frac{2\pi R}{R-|\alpha|} \max_{|z|=R} |f(z)| \quad \text{when } R > |\alpha|\,.$$

**Exercise 3.7.** Suppose $f : \mathbb{C} \to \mathbb{C}$ is a continuous function such that:

$$\lim_{R\to+\infty} \sup_{|z|\geq R} \frac{|f(z)|}{R} = 0.$$

Show that:

$$\lim_{R\to+\infty} \oint_{|z|=R} \frac{f(z)}{z^2}\, dz = 0.$$

**Exercise 3.8.** Let $f : \mathbb{C} \to \mathbb{R}$ be a continuous *real-valued* function such that $|f(z)| \leq 1$ for any $z \in \mathbb{C}$. Show that:

$$\left| \oint_{|z|=1} f(z)\, dz \right| \leq 4.$$

[Hint: Define $I = \oint_{|z|=1} f(z)\, dz$, then write $I = |I|\, e^{i\theta}$. ]

## 3.2. Cauchy-Goursat's Theorem

In this section, we will prove a very fundamental theorem in Complex Analysis, the Cauchy-Goursat's Theorem, which asserts that if $f : \Omega \to \mathbb{C}$ is a holomorphic function on a simply-connected domain $\Omega$, then the contour integral $\oint_\gamma f(z)\,dz$ must be zero for any closed curve $\gamma$ in $\Omega$. The statement of the theorem sounds simple, but the proof is quite delicate. We will discuss the proof of this theorem in detail.

Cauchy-Goursat's Theorem is fundamental because it is used to prove the Cauchy's integral formula, which provides a very elegant way for computing contour integral of the form $\oint_\gamma \frac{f(z)}{z-\alpha}\,dz$ and leading many exciting results. We will see later in the course that the Cauchy integral formula is the *heart* of complex analysis.

**Theorem 3.8** (Cauchy-Goursat's Theorem). *Let $\Omega \subset \mathbb{C}$ be a simply-connected open domain, $\gamma$ be any closed piecewise $C^1$ curve in $\Omega$, and $f : \Omega \to \mathbb{C}$ be any holomorphic function defined on $\Omega$, then we have:*

$$\oint_\gamma f(z)\,dz = 0.$$

Using Cauchy-Goursat's Theorem, we can immediately conclude that all the integrals below over any closed curve $\gamma \in \mathbb{C}$ are zero, without performing any calculation:

$$\oint_\gamma e^z\,dz, \quad \oint_\gamma \sin z\,dz, \quad \oint_\gamma z^2\,dz, \quad \text{etc.}$$

Both conditions of $\Omega$ being simply-connected and $f$ being holomorphic on $\Omega$ are essential. If $\Omega$ is not simply-connected, say $\Omega = \mathbb{C}\backslash\{0\}$, Cauchy-Goursat's Theorem does not hold. Here is a quick counter-example:

$$\oint_{|z|=1} \frac{1}{z}\,dz = 2\pi i \neq 0.$$

Moreover, the holomorphic condition on $f$ is also necessary, and here is a counter-example:

$$\oint_{|z|=1} \bar{z}\,dz = 2\pi i \neq 0.$$

We will prove this theorem soon. The proof consists of several steps:

**Step 1:** First prove a special case when the contour $\gamma$ is a *triangle* (while $\Omega$ is any simply-connected open domain);

**Step 2:** Then prove a special case when $\Omega$ is *convex* (while $\gamma$ is any closed piecewise $C^1$ contour).

**Step 3:** Use results from previous steps to deduce the general case: $\Omega$ is any simply-connected open domain, and $\gamma$ is any closed piecewise $C^1$ contour.

**3.2.1. Step 1: Cauchy-Goursat's Theorem for Triangle Contours.** Let's begin by assuming that $T$ is a triangle contour in $\Omega$. We bisect each side of the triangle $T$ to create four smaller triangles $T_1^{(1)}$, $T_2^{(1)}$, $T_3^{(1)}$ and $T_4^{(1)}$ as shown in the Figure 3.2.

By cancellations of common sides, we have:

$$\oint_T f(z)\,dz = \sum_{j=1}^{4} \oint_{T_j^{(1)}} f(z)\,dz.$$

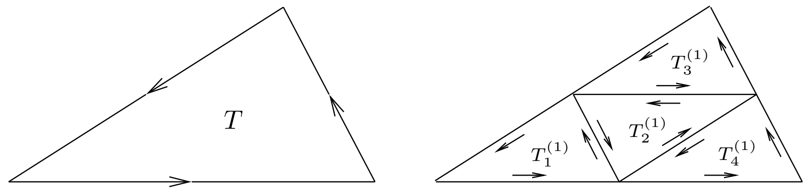



**Figure 3.2.** Divide the contour $T$ into 4 triangles

Triangle inequality then shows:

$$\left|\oint_T f(z)\,dz\right| \leq \sum_{j=1}^{4} \left|\oint_{T_j^{(1)}} f(z)\,dz\right|.$$

Let $T^{(1)}$ be the triangle among all $T_j^{(1)}$'s (where $j = 1, 2, 3, 4$) with the largest value of $\left|\oint_{T_j^{(1)}} f(z)\,dz\right|$, then one has:

$$\left|\oint_T f(z)\,dz\right| \leq 4\left|\oint_{T^{(1)}} f(z)\,dz\right|$$

Repeat the above procedure on $T^{(1)}$: sub-divide $T^{(1)}$ into four congruent triangles $T_j^{(2)}$ (where $j = 1, 2, 3, 4$), and pick the one with the largest value of $\left|\oint_{T_j^{(2)}} f(z)\,dz\right|$ and label it as $T^{(2)}$. Then, one has:

$$\left|\oint_{T^{(1)}} f(z)\,dz\right| \leq 4\left|\oint_{T^{(2)}} f(z)\,dz\right| \quad \Longrightarrow \quad \left|\oint_T f(z)\,dz\right| \leq 4^2\left|\oint_{T^{(2)}} f(z)\,dz\right|.$$

Continuing this process, we obtain a sequence of triangles:

$$T^{(0)}, T^{(1)}, T^{(2)}, T^{(3)}, \ldots$$

(where we denote $T^{(0)} := T$) such that

$$\left|\oint_{T^{(0)}} f(z)\,dz\right| \leq 4^n\left|\oint_{T^{(n)}} f(z)\,dz\right| \quad \text{for any } n \geq 0. \tag{3.1}$$

Denote $\Delta^{(j)}$ to be the closed triangular region enclosed by $T^{(j)}$. Then, we have:

$$\Delta^{(0)} \supset \Delta^{(1)} \supset \Delta^{(2)} \supset \ldots$$

By Exercise 1.29, there is at least one point $z_0$ contained inside all of $\Delta^{(n)}$.

Our goal is to bound the RHS term $4^n\left|\oint_{T^{(n)}} f(z)\,dz\right|$ of (3.1), so as to show that $\left|\oint_{T^{(0)}} f(z)\,dz\right|$ is arbitrarily small, concluding that it must be zero. To achieve our goal, we recall that $f$ is holomorphic on $\Omega$, and in particular, it is complex differentiable at $z_0$ (which is a point in all of $\Delta^{(n)}$'s). By considering the derivative $f'(z_0)$, and by rearrangement:

$$f'(z_0) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} \quad \Longrightarrow \quad \lim_{z \to z_0} \frac{f(z) - f(z_0) - f'(z_0)(z - z_0)}{z - z_0} = 0.$$

For simplicity, denote the numerator by $E(z) := f(z) - f(z_0) - f'(z_0)(z - z_0)$, then we have:

$$\lim_{z \to z_0} \frac{E(z)}{z - z_0} = 0. \tag{3.2}$$

Since the function $f(z_0) + f'(z_0)(z - z_0)$ has a primitive function $zf(z_0) + \frac{f'(z_0)}{2}(z - z_0)^2$ (note that $z_0$ is a fixed point), we have

$$\oint_{T^{(n)}} E(z)\,dz = \oint_{T^{(n)}} \left[f(z) - f(z_0) - f'(z_0)(z - z_0)\right]\,dz = \oint_{T^{(n)}} f(z)\,dz.$$

Therefore, to bound the RHS of (3.1), we can consider the integral of $E(z)$ instead, which is very small according to (3.2).

Now, given any $\varepsilon > 0$, by (3.2), there exists $\delta > 0$ such that whenever $z \in B_\delta(z_0)$, we have $\left|\frac{E(z)}{z - z_0}\right| < \varepsilon$. Recall that $\{\Delta^{(n)}\}_{n=0}^{\infty}$ is a strictly decreasing sequence of triangles "converging" to the point $z_0$. Hence, for sufficiently large $n$, $\Delta^{(n)}$ must lie inside the ball $B_\delta(z_0)$, and so $|E(z)| < \varepsilon\,|z - z_0|$ for any $z \in \Delta^{(n)} \subset B_\delta(z_0)$.

Recall that $|z - z_0|$ is the distance between $z$ and $z_0$, both of which are in $\Delta^{(n)}$. By elementary geometry, the distance between any two points in a triangle must be bounded by the perimeter of the triangle. Hence, we have for any $z \in \Delta^{(n)}$,

$$|E(z)| < \varepsilon\,|z - z_0| \leq \varepsilon L_n = \frac{\varepsilon L_0}{2^n} \tag{3.3}$$

where $L_n$ denotes the perimeter of the triangle $T^{(n)}$.

Using (3.3), we can apply Lemma 3.6 to show:

$$\left|\oint_{T^{(n)}} E(z)\,dz\right| \leq \frac{\varepsilon L_0}{2^n} \cdot L_n = \frac{\varepsilon L_0^2}{4^n}.$$

Finally, by considering (3.1), we have proved:

$$\left|\oint_T f(z)\,dz\right| \leq 4^n \left|\oint_{T^{(n)}} f(z)\,dz\right| = 4^n \left|\oint_{T^{(n)}} E(z)\,dz\right| \leq 4^n \cdot \frac{\varepsilon L_0^2}{4^n} = \varepsilon L_0^2.$$

Since $\varepsilon > 0$ is arbitrary, by letting $\varepsilon \to 0^+$, we get:

$$\oint_T f(z)\,dz = 0,$$

completing Step 1.

**Exercise 3.9.** Using the result proved so far, show that Cauchy-Goursat's Theorem holds for any closed *polygon* $\gamma$.

**Exercise 3.10.** Show that if $\triangle ABC$ is contained inside a simply-connected open set $\Omega$ on which $f$ is holomorphic, then we have:

$$\int_{L(A,C)} f(z)\,dz = \int_{L(A,B)} f(z)\,dz + \int_{L(B,C)} f(z)\,dz.$$

Here $L(A,B)$, for instance, is the straight path from $A$ to $B$.

**Exercise 3.11.** Which part in the proof of Step 1 will break down if $f$ is not holomorphic? Also, why will the proof break down if $\Omega$ is not simply-connected?

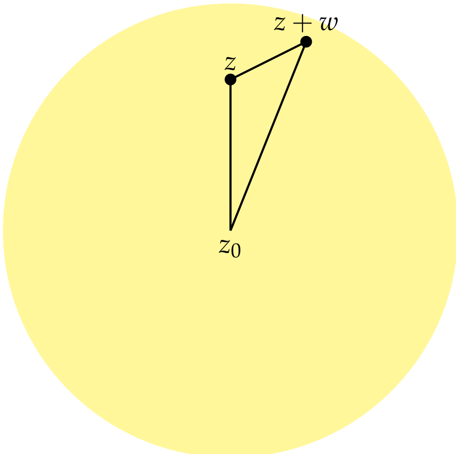
**3.2.2. Step 2: Cauchy-Goursat's Theorem for Convex Domains.** Now we are given any closed piecewise $C^1$ curve $\gamma$ (not necessarily a triangle) in an open convex domain $\Omega$. We want to show that if $f : \Omega \to \mathbb{C}$ is holomorphic, then

$$\oint_\gamma f(z)\,dz = 0.$$

We show that by finding a primitive function $F : \Omega \to \mathbb{C}$ such that $F'(z) = f(z)$ on $\Omega$, then this step is proved using Theorem 3.4. To define such a function $F$, we first fix a point $z_0 \in \Omega$, and denote $L(z_0, z)$ to be the straight path from $z_0$ to $z$. Note that by convexity of $\Omega$, such a path must be contained in $\Omega$. Next, we define:

$$F(z) := \int_{L(z_0,z)} f(\xi)\,d\xi.$$

We claim that $F'(z) = f(z)$ by showing that the quotient $\dfrac{F(z+w) - F(z)}{w}$ tends to $f(z)$ as $w \to 0$.




From Step 1 (note that $z_0$, $z$ and $z+w$ form a triangle), we know that:

$$\begin{aligned}\frac{F(z+w) - F(z)}{w} &= \frac{1}{w}\left(\int_{L(z_0,z+w)} f(\xi)\,d\xi - \int_{L(z_0,z)} f(\xi)\,d\xi\right) \\ &= \frac{1}{w}\int_{L(z,z+w)} f(\xi)\,d\xi.\end{aligned}$$

By observing that $\int_{L(z,z+w)} f(z)\,d\xi = [f(z)\,\xi]_{\xi=z}^{\xi=z+w} = w\,f(z)$, we have:

$$\begin{aligned}\frac{F(z+w) - F(z)}{w} &= \frac{1}{w}\int_{L(z,z+w)} f(\xi)\,d\xi = \frac{1}{w}\int_{L(z,z+w)} (f(\xi) - f(z)) + f(z)\,d\xi \\ &= \frac{1}{w}\int_{L(z,z+w)} (f(\xi) - f(z))\,d\xi + f(z).\end{aligned} \tag{3.4}$$

The next task will be to show that $\dfrac{1}{w}\displaystyle\int_{L(z,z+w)} (f(\xi) - f(z))\,d\xi$ tends to 0 as $w \to 0$. For any $\varepsilon > 0$, by the continuity of $f$, there exists $\delta > 0$ such that whenever $\xi \in B_\delta(z)$, we have $|f(\xi) - f(z)| < \varepsilon$. In particular, if $|w| < \delta$, then the path $L(z, z+w) \subset B_\delta(z)$, and so for any $\xi \in L(z, z+w)$, we have:

$$|f(\xi) - f(z)| < \varepsilon.$$

Applying Lemma 3.6 on the integral $\int_{L(z,z+w)} (f(\xi) - f(z))\, d\xi$, we have:

$$\left| \int_{L(z,z+w)} (f(\xi) - f(z))\, d\xi \right| \leq \varepsilon \cdot \underbrace{|w|}_{\text{length of contour}},$$

which implies $\left| \frac{1}{w} \int_{L(z,z+w)} (f(\xi) - f(z))\, d\xi \right| \leq \varepsilon$ (whenever $0 < |w| < \delta$), or equivalently,

$$\lim_{w \to 0} \frac{1}{w} \int_{L(z,z+w)} (f(\xi) - f(z))\, d\xi = 0.$$

Finally, from (3.4), we conclude:

$$\lim_{w \to 0} \frac{F(z+w) - F(z)}{w} = f(z) \quad \Longrightarrow \quad F'(z) = f(z).$$

This shows $f(z)$ has a primitive function on $\Omega$, and hence

$$\oint_\gamma f(z)\, dz = 0$$

for any closed curve $\gamma$ in $\Omega$, completing Step 2.

**Remark 3.9.** It is worthwhile to note that the whole argument in Step 2 remains valid as long as $f$ is continuous on $\Omega$, and that

$$\oint_T f(z)\, dz = 0$$

for any triangle $T$ in the domain $\Omega$. These two conditions are enough to prove, using the same argument, that $F'(z) = f(z)$ on $\Omega$, even if we don't assume $f$ is holomorphic. This observation will be important in the proof of Morera's Theorem in later section.

**Exercise 3.12.** Discuss: In the above proof, we require $\Omega$ to be convex so that $L(z_0, z)$ is contained in $\Omega$. Now suppose $\Omega$ is not convex, but is polygonally path-connected, and we define $F$ as:

$$F(z) = \int_{\gamma(z_0,z)} f(\xi)\, d\xi$$

where $\gamma(z_0, z)$ is *any* polygonal path from $z_0$ to $z$. Can we still claim that $F'(z) = f(z)$ with the same proof? If not, where does the proof break down?

#### 3.2.3. Step 3: Completion of the Proof.

We have by far proved that Cauchy-Goursat's Theorem holds when at least one of the conditions holds:

(i) $\gamma$ is a closed polygon; or
(ii) $\Omega$ is convex.

Now we deduce the general case based on these special cases.

Given any simply-connected domain $\Omega$ and any closed piecewise $C^1$ curve $\gamma \subset \Omega$, and a holomorphic function $f : \Omega \to \mathbb{C}$, the key idea to show $\oint_\gamma f(z)\, dz = 0$ is to break the region enclosed by $\gamma$ into small rectangles $\{R_j\}_{j=1}^N$ and "partial rectangles" $\{\gamma_k\}_{k=1}^M$ (see Figure 3.3). By breaking the region into small enough of these rectangles and partial rectangles, we may assume that these partial rectangles are contained inside an convex subset of $\Omega$. This is intuitively true, but the proof involves some deep knowledge on analysis and topology beyond the scope of this course.

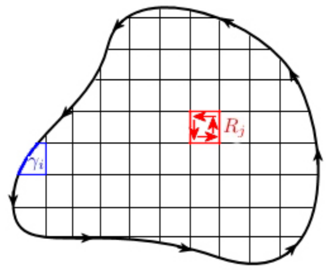



**Figure 3.3**

For each rectangle $R_j$ and partial rectangle $\gamma_k$, results from Steps 1 and 2 show

$$\oint_{R_j} f(z)\,dz = \oint_{\gamma_k} f(z)\,dz = 0.$$

Note that by cancellation of common sides, we can see:

$$\oint_{\gamma} f(z)\,dz = \sum_j \oint_{R_j} f(z)\,dz + \sum_k \oint_{\gamma_k} f(z)\,dz = 0.$$

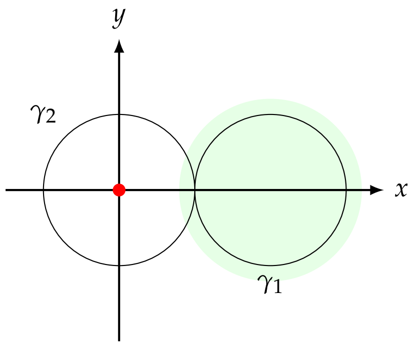
It completes the proof of Cauchy-Goursat's Theorem.

**Exercise 3.13.** Consider a holomorphic $f = u + iv : \Omega \to \mathbb{C}$ on a simply-connected domain $\Omega$, and a closed piecewise $C^1$ curve $\gamma$ in $\Omega$. Now, we *further assume* that $f$ is $C^1$, i.e. $\frac{\partial u}{\partial x}$, $\frac{\partial u}{\partial y}$, $\frac{\partial v}{\partial x}$ and $\frac{\partial v}{\partial y}$ are all continuous on $C^1$, show that then Cauchy-Goursat's Theorem can be *easily* proved using Green's Theorem.

## 3.3. Cauchy's Integral Formula I

Cauchy-Goursat's Theorem requires that the function $f$ involved is defined and holomorphic in the region enclosed by the closed curve $\gamma$. When the integrand has some "singularities" such as $f(z) = \frac{1}{z}$, Cauchy-Goursat's Theorem may not hold.

Consider the closed curves $\gamma_1$ and $\gamma_2$ shown below:




For $\gamma_1$, there is no issue to apply Cauchy-Goursat's Theorem by taking $\Omega$ to be the green region, and it shows

$$\oint_{\gamma_1} \frac{1}{z}\, dz = 0$$

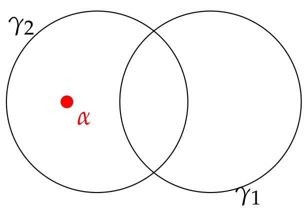
since $\frac{1}{z}$ is holomorphic on the green region. However, we cannot do the same for $\gamma_2$. Any simply-connected region containing $\gamma_2$ must contain 0 at which $\frac{1}{z}$ is undefined. In this section, we will introduce Cauchy's integral formula to deal with contour integrals of the form $\oint_\gamma \frac{f(z)}{z-\alpha}\, dz$.

**Theorem 3.10** (Cauchy's Integral Formula)**.** *Let $f : \Omega \to \mathbb{C}$ be a holomorphic function defined on a simply-connected domain $\Omega$, and $\gamma$ be a simple closed curve in $\Omega$. Then, we have:*

$$\frac{1}{2\pi i} \oint_\gamma \frac{f(z)}{z-\alpha}\, dz = \begin{cases} f(\alpha) & \text{if } \gamma \text{ encloses } \alpha \\ 0 & \text{if } \gamma \text{ does not enclose } \alpha \end{cases}$$

For instance, given an entire function $f : \mathbb{C} \to \mathbb{C}$, a point $\alpha$, and two closed curves $\gamma_1$ and $\gamma_2$ below. Cauchy's Integral Formula asserts that:

$$\oint_{\gamma_1} \frac{f(z)}{z-\alpha}\, dz = 0 \quad \text{whereas} \quad \oint_{\gamma_2} \frac{f(z)}{z-\alpha}\, dz = 2\pi i f(\alpha).$$




It is a very powerful theorem as it tells us that the evaluation of some contour integrals can be done by just substituting a point into the numerator function. Let's first see some examples, and then we will prove the theorem.

**3.3.1. Elementary Examples.** We first illustrate the use of Cauchy's integral formula by a toy example:

$$\oint_\gamma \frac{1}{z}\,dz = \oint_\gamma \frac{1}{z-0}\,dz = \begin{cases} 2\pi i \cdot 1 = 2\pi i & \text{if } \gamma \text{ encloses } 0 \\ 0 & \text{if } \gamma \text{ does not enclose } 0 \end{cases}$$

Here we take $f(z) = 1$ which is an entire function on $\mathbb{C}$.

**Example 3.5.** Evaluate the following contour integrals:

(a) $\displaystyle\oint_{|z|=2} \frac{z}{(z+3i)(z-i)}\,dz$

(b) $\displaystyle\oint_{|z|=4} \frac{z}{(z+3i)(z-i)}\,dz$

(c) $\displaystyle\oint_{|z|=2} \frac{e^z}{z^2+1}\,dz$

**Solution**

(a) The integrand has two singularities: $z = -3i$ and $z = i$. First observe that the curve $|z| = 2$ enclose $i$ only, and hence near the simply-connected region $|z| \leq 2$, the function $f(z) := \frac{z}{z+3i}$ is holomorphic. Apply Cauchy's integral formula with this $f$, we get:

$$\oint_{|z|=2} \frac{z}{(z+3i)(z-i)}\,dz = \oint_{|z|=2} \frac{\frac{z}{z+3i}}{z-i}\,dz = 2\pi i \cdot \left.\frac{z}{z+3i}\right|_{z=i}$$
$$= 2\pi i \cdot \frac{i}{i+3i} = \frac{\pi i}{2}.$$

(b) Note that the curve $|z| = 4$ enclose both singularities $-3i$ and $i$ of the integrand. We cannot apply Cauchy's integral formula by writing the integrand as either:

$$\frac{\frac{z}{z+3i}}{z-i} \quad \text{or} \quad \frac{\frac{z}{z-i}}{z+3i}.$$

The way out is to do partial fractions for the denominator. Let $A$ and $B$ be complex numbers such that:

$$\frac{1}{(z+3i)(z-i)} = \frac{A}{z+3i} + \frac{B}{z-i}.$$

We need to solve for $A$ and $B$:

$$\frac{1}{(z+3i)(z-i)} = \frac{A(z-i)+B(z+3i)}{(z+3i)(z-i)}$$
$$1 = (A+B)z + (-Ai+3Bi)$$

Equating coefficients, we need $A + B = 0$ and $(-Ai + 3Bi) = 1$. Solving these equations, we get $A = \frac{1}{4}i$ and $B = -\frac{1}{4}i$, and hence:

$$\frac{1}{(z+3i)(z-i)} = \frac{\frac{1}{4}i}{z+3i} - \frac{\frac{1}{4}i}{z-i}.$$

Now applying Cauchy's integral formula:

$$\begin{aligned}\oint_{|z|=4} \frac{z}{(z+3i)(z-i)}\,dz &= \oint_{|z|=4} \frac{\frac{1}{4}zi}{z+3i} - \frac{\frac{1}{4}zi}{z-i}\,dz \\ &= 2\pi i\left(\left[\frac{1}{4}zi\right]_{z=-3i} - \left[\frac{1}{4}zi\right]_{z=i}\right) \\ &= 2\pi i\left(\frac{1}{4}\cdot(-3i)i - \frac{1}{4}i^2\right) = 2\pi i.\end{aligned}$$

(c) The integrand has $z^2+1$ as the denominator. Be careful that it can be zero in the complex world and so $\dfrac{e^z}{z^2+1}$ is NOT holomorphic everywhere. By partial fractions, we get:

$$\frac{1}{z^2+1} = \frac{1}{(z-i)(z+i)} = \frac{1}{2i}\left(\frac{1}{z-i} - \frac{1}{z+i}\right).$$

Hence, Cauchy's integral formula shows:

$$\begin{aligned}\oint_{|z|=2} \frac{e^z}{z^2+1}\,dz &= \frac{1}{2i}\oint_{|z|=2}\left(\frac{e^z}{z-i} - \frac{e^z}{z+i}\right)dz \\ &= \frac{1}{2i}\cdot 2\pi i\cdot\left(e^i - e^{-i}\right) \\ &= \pi\left((\cos 1 + i\sin 1) - (\cos 1 - i\sin 1)\right) \\ &= 2\pi i\sin 1.\end{aligned}$$

**Exercise 3.14.** Use Cauchy's integral formula to evaluate the following contour integrals:

(a) $\displaystyle\oint_{|z|=2} \frac{1}{z^2+i}\,dz$

(b) $\displaystyle\oint_{|z-e^{\pi i/4}|=1} \frac{1}{z^2+i}\,dz$

(c) $\displaystyle\oint_{|z|=2} \frac{1}{z^3-1}\,dz$

Try to do the problems in a rather *tedious* way using partial fractions. We will provide another approach soon.

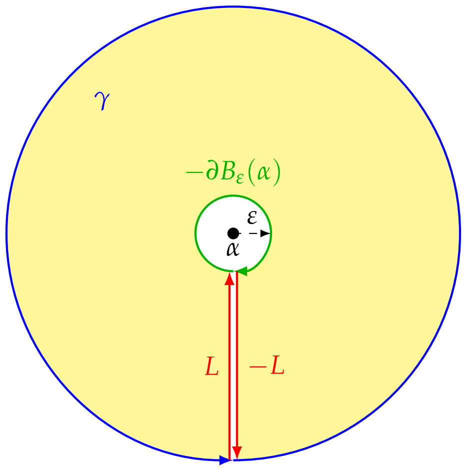
**Exercise 3.15.** Let $f:\Omega\to\mathbb{C}$ be a holomorphic function defined on a domain $\Omega$ containing $B_r(\alpha)$. Prove the following Mean-Value Identity:

$$f(\alpha) = \frac{1}{2\pi}\int_0^{2\pi} f(\alpha + re^{i\theta})\,d\theta.$$

**3.3.2. Proof of Cauchy's Integral Formula.** The proof of Cauchy's integral formula is a reminiscence of the proof of generalized (i.e. with holes) Green's Theorem in Multivariable Calculus. Fix $\alpha\in\mathbb{C}$ and consider a simple closed curve $\gamma$ enclosing $\alpha$. We want to find out the value of the integral:

$$\oint_\gamma \frac{f(z)}{z-\alpha}\,dz.$$

We drill a circular hole near $\alpha$ in the region enclosed by $\gamma$, so that the following "key-hole" contour $\Gamma_\varepsilon$ is produced.




The contour $\Gamma_\varepsilon = \gamma + L - \partial B_\varepsilon(\alpha) - L$ encloses a simply-connected region on which $\dfrac{f(z)}{z-\alpha}$ is holomorphic (since $z \neq \alpha$ in this key-hole region). Therefore, we have:

$$0 = \oint_{\Gamma_\varepsilon} \frac{f(z)}{z-\alpha}\,dz = \oint_\gamma \frac{f(z)}{z-\alpha}\,dz + \oint_L \frac{f(z)}{z-\alpha} \underbrace{- \oint_{|z-\alpha|=\varepsilon} \frac{f(z)}{z-\alpha}\,dz}_{\text{orientation!}} - \oint_L \frac{f(z)}{z-\alpha}$$
$$= \oint_\gamma \frac{f(z)}{z-\alpha}\,dz - \oint_{|z-\alpha|=\varepsilon} \frac{f(z)}{z-\alpha}\,dz.$$

Therefore, we have $\oint_\gamma \frac{f(z)}{z-\alpha}\,dz = \oint_{|z-\alpha|=\varepsilon} \frac{f(z)}{z-\alpha}\,dz$ for any sufficiently small $\varepsilon > 0$. To prove the desired result, we try to figure out the contour integral over the circle $|z-\alpha| = \varepsilon$. The key trick is to write $f(z) = f(z) - f(\alpha) + f(\alpha)$, so that:

$$\oint_{|z-\alpha|=\varepsilon} \frac{f(z)}{z-\alpha}\,dz = \oint_{|z-\alpha|=\varepsilon} \left( \frac{f(z)-f(\alpha)}{z-\alpha} + \frac{f(\alpha)}{z-\alpha} \right) dz$$
$$= \oint_{|z-\alpha|=\varepsilon} \frac{f(z)-f(\alpha)}{z-\alpha}\,dz + f(\alpha) \oint_{|z-\alpha|=\varepsilon} \frac{1}{z-\alpha}\,dz \tag{3.5}$$

The second integral can be computed directly by parametrizing the circle: $z = \alpha + \varepsilon e^{it}$, where $t \in [0, 2\pi]$:

$$\oint_{|z-\alpha|=\varepsilon} \frac{1}{z-\alpha}\,dz = \int_0^{2\pi} \frac{1}{\varepsilon e^{it}} \cdot \varepsilon i e^{it}\,dt$$
$$= \int_0^{2\pi} i\,dt = 2\pi i.$$

For the first term, we claim that it tends to 0 as $\varepsilon \to 0^+$: since $f$ is complex differentiable at $z = \alpha$, and so

$$\lim_{z\to\alpha} \frac{f(z)-f(\alpha)}{z-\alpha} = f'(\alpha).$$

By definition of limit, there exists $\delta > 0$ such that whenever $z \in B_\delta(\alpha)$ we have:

$$\left| \frac{f(z)-f(\alpha)}{z-\alpha} - f'(\alpha) \right| < 1,$$

and hence

$$\left| \frac{f(z)-f(\alpha)}{z-\alpha} \right| < 1 + |f'(\alpha)| =: M.$$

As a result, when $\varepsilon < \delta$, the contour $|z-\alpha| = \varepsilon$ lies completely inside the ball $B_\delta(\alpha)$, then by Lemma 3.6, we have:

$$\left| \oint_{|z-\alpha|=\varepsilon} \frac{f(z) - f(\alpha)}{z-\alpha} \right| \leq M \cdot 2\pi\varepsilon \to 0 \text{ as } \varepsilon \to 0^+.$$

Finally, from (3.5), we have:

$$\lim_{\varepsilon \to 0^+} \oint_{|z-\alpha|=\varepsilon} \frac{f(z)}{z-\alpha}\, dz = 2\pi i f(\alpha).$$

Recall that $\oint_\gamma \frac{f(z)}{z-\alpha}\, dz = \oint_{|z-\alpha|=\varepsilon} \frac{f(z)}{z-\alpha}\, dz$ for any sufficiently small $\varepsilon > 0$, so we have:

$$\oint_\gamma \frac{f(z)}{z-\alpha}\, dz = \lim_{\varepsilon \to 0^+} \oint_{|z-\alpha|=\varepsilon} \frac{f(z)}{z-\alpha}\, dz = 2\pi i f(\alpha),$$

completing the proof of Cauchy's integral formula.

**3.3.3. Cauchy's Integral Formula with Multiple Holes.** We have seen how to apply Cauchy's integral formula on fractions such as $\frac{1}{z^2+1}$ which is not defined on $z = i$ and $z = -i$. If a simple closed contour $\gamma$ encloses both singularities, then we performed partial fractions so that the fraction becomes $\frac{1}{2i}\left(\frac{1}{z-i} - \frac{1}{z+i}\right)$.

Sometimes, partial fractions can be time-consuming especially when there are many singularities. However, using the hole-drilling technique demonstrated in the proof of Cauchy's integral formula, we can break down the contour integral into a sum of several contour integrals, each of which is over a contour that encloses only *one* singularity. Let's look at some examples.

**Example 3.6.** Evaluate the contour integral:

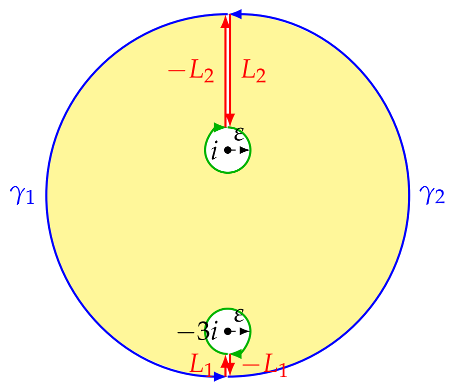
$$\oint_{|z|=4} \frac{z}{(z+3i)(z-i)}\, dz$$

without using partial fractions.

**Solution**

The two singularities are $z = -3i$ and $z = i$, both are contained inside the contour $|z| = 4$. Draw two little circles with small radii $\varepsilon$ around each singularity and consider the key-hole contour:

$$\Gamma = \gamma_1 + L_1 - \partial B_\varepsilon(-3i) - L_1 + \gamma_2 + L_2 - \partial B_\varepsilon(i) - L_2$$



Then, the key hole contour $\Gamma$ encloses a simply-connected region not containing any singularity of the integrand. Therefore, Cauchy-Goursat's Theorem asserts that

$$\oint_\Gamma \frac{z}{(z+3i)(z-i)}\,dz = 0.$$

On the other hand, by cancellation of the common sides, we have:

$$\oint_\Gamma = \int_{\gamma_1} + \int_{\gamma_2} - \oint_{|z+3i|=\varepsilon} - \oint_{|z-i|=\varepsilon} = \oint_{|z|=4} - \oint_{|z+3i|=\varepsilon} - \oint_{|z-i|=\varepsilon}.$$

Therefore,

$$\begin{aligned} 0 &= \oint_\Gamma \frac{z}{(z+3i)(z-i)}\,dz \\ &= \oint_{|z|=4} \frac{z}{(z+3i)(z-i)}\,dz - \oint_{|z+3i|=\varepsilon} \frac{z}{(z+3i)(z-i)}\,dz \\ &\quad - \oint_{|z-i|=\varepsilon} \frac{z}{(z+3i)(z-i)}\,dz. \end{aligned}$$

Therefore, we can break the required integral into the sum of two integrals:

$$\oint_{|z|=4} \frac{z}{(z+3i)(z-i)}\,dz = \oint_{|z+3i|=\varepsilon} \frac{z}{(z+3i)(z-i)}\,dz + \oint_{|z-i|=\varepsilon} \frac{z}{(z+3i)(z-i)}\,dz$$

Since $\varepsilon$ is very small, the function $\dfrac{z}{z-i}$ is holomorphic on $|z+2i| < \varepsilon$, and so Cauchy's integral formula asserts that:

$$\oint_{|z+3i|=\varepsilon} \frac{z}{(z+3i)(z-i)}\,dz = \oint_{|z+3i|=\varepsilon} \frac{\frac{z}{z-i}}{z-(-3i)}\,dz = 2\pi i \cdot \frac{-3i}{-3i-i} = \frac{3\pi i}{2}.$$

For the second integral, we have:

$$\oint_{|z-i|=\varepsilon} \frac{z}{(z+3i)(z-i)}\,dz = \oint_{|z-i|=\varepsilon} \frac{\frac{z}{z+3i}}{z-i}\,dz = 2\pi i \cdot \frac{i}{i+3i} = \frac{\pi i}{2}$$

Adding up the results, we get:

$$\oint_{|z|=4} \frac{z}{(z+3i)(z-i)}\,dz = \frac{3\pi i}{2} + \frac{\pi i}{2} = 2\pi i.$$

**Example 3.7.** Evaluate the contour integral:

$$\oint_{|z|=2} \frac{1}{z^3-1}\,dz$$

without using partial fractions.

**Solution**

First factorize the integrand:

$$\frac{1}{z^3-1} = \frac{1}{(z-1)(z-\omega)(z-\omega^2)}$$

where $\omega := e^{\frac{2\pi i}{3}}$ is the cubic root of unity. There are three singularities, namely $1$, $\omega$ and $\omega^2$, all are enclosed by the given contour $|z| = 2$. By mimicking the hole-drilling argument, one can arrive at:

$$\begin{aligned}
&\oint_{|z|=2} \frac{1}{(z-1)(z-\omega)(z-\omega^2)}\,dz \\
&= \oint_{|z-1|=\varepsilon} \frac{1}{(z-1)(z-\omega)(z-\omega^2)}\,dz + \oint_{|z-\omega|=\varepsilon} \frac{1}{(z-1)(z-\omega)(z-\omega^2)}\,dz \\
&\quad + \oint_{|z-\omega^2|=\varepsilon} \frac{1}{(z-1)(z-\omega)(z-\omega^2)}\,dz \\
&= \oint_{|z-1|=\varepsilon} \frac{\frac{1}{(z-\omega)(z-\omega^2)}}{z-1}\,dz + \oint_{|z-\omega|=\varepsilon} \frac{\frac{1}{(z-1)(z-\omega^2)}}{z-\omega}\,dz + \oint_{|z-\omega^2|=\varepsilon} \frac{\frac{1}{(z-1)(z-\omega)}}{z-\omega^2}\,dz \\
&= 2\pi i \left[\frac{1}{(1-\omega)(1-\omega^2)} + \frac{1}{(\omega-1)(\omega-\omega^2)} + \frac{1}{(\omega^2-1)(\omega^2-\omega)}\right].
\end{aligned}$$

We leave it as an exercise to show that the final answer is 0. [Hint: use the fact that $1+\omega+\omega^2=0$]

**Exercise 3.16.** Evaluate the following contour integrals:

(a) $\displaystyle\oint_{|z|=24601} \frac{1}{z^3+1}\,dz$

(b) $\displaystyle\oint_{|z|=2} \frac{1}{(z^2+1)(z^2+9)}\,dz$

(c) $\displaystyle\oint_{|z-1|=1} \frac{e^z}{z^4+1}\,dz$

(d) $\displaystyle\oint_{|z|=4} \frac{z}{1-e^z}\,dz$

**Exercise 3.17.** Let $n$ be a positive integer, and $\omega := e^{2\pi i/n}$ denote the $n$-th root of unity. Express the contour integral:

$$\oint_{|z|=2} \frac{1}{z^n-1}\,dz$$

in terms of $\omega$.

**Exercise 3.18.** Given any real constant $a \in \mathbb{R}$, by considering the contour integral $\displaystyle\oint_{|z|=1} \frac{e^{az}}{z}\,dz$, prove the following integration formula:

$$\int_0^{\pi} e^{a\cos\theta}\cos(a\sin\theta)\,d\theta = \pi.$$

## 3.4. Cauchy's Integral Formula II

Recall that Cauchy's integral formula asserts that if $f : \Omega \to \mathbb{C}$ is holomorphic on a simply-connected domain $\Omega$ and $\gamma$ is a closed curve in $\Omega$, then we have:

$$f(\alpha) = \frac{1}{2\pi i} \oint_\gamma \frac{f(z)}{z-\alpha}\, dz$$

if $\gamma$ encloses $\alpha$.

If the integrand is of the form $\dfrac{f(z)}{(z-\alpha)(z-\beta)}$ whenever $\alpha \neq \beta$, we can still use Cauchy's integral formula in a modified way: either by partial fractions, or by a hole-drilling argument illustrated in the previous section.

However, if the integrand is of the form $\dfrac{f(z)}{(z-\alpha)^2}$, then both partial fractions and the hole-drilling argument do not work well (think about why). Indeed, the contour integral $\oint_\gamma \dfrac{f(z)}{(z-\alpha)^2}\, dz$ is related to $f'(\alpha)$, and this fact has many deep and surprising consequences as we will see later. These include the celebrated Liouville's Theorem (which implies Fundamental Theorem of Algebra).

Our goal is to prove and discuss the following higher-order Cauchy's integral formula:

**Theorem 3.11** (Higher-Order Cauchy's Integral Formula)**.** *Let $f : \Omega \to \mathbb{C}$ be a holomorphic function defined on a simply-connected domain $\Omega$, and $\alpha$ be any point in $\Omega$. Then, for any simple closed curve $\gamma$ enclosing $\alpha$, the $n$-th derivative of $f$ at $\alpha$ is equal to:*

$$f^{(n)}(\alpha) = \frac{n!}{2\pi i} \oint_\gamma \frac{f(z)}{(z-\alpha)^{n+1}}\, dz, \quad n = 0, 1, 2, \ldots$$

**Corollary 3.12.** *If $f$ is holomorphic on an open domain $\Omega$, then $f$ is complex differentiable for infinitely many times on $\Omega$, i.e. $f^{(n)}$ exists on $\Omega$ for any $n \geq 0$.*

The corollary is a very remarkable and surprising result. In Real Analysis, there are many functions which are differentiable for one time but not the second time or further. However, this theorem and the corollary assert that once $f$ is *complex* differentiable on a simply-connected domain (say an open ball), then it is infinitely differentiable on that domain!

**3.4.1. Elementary Examples.** Again, we will first see some examples of using the higher-order Cauchy's integral formula, then we will give a proof for it. As a quick example:

$$\oint_{|z|=1} \frac{1}{z^2}\, dz.$$

One way of evaluating it is to argue that its primitive function is $-\frac{1}{z}$, which is well defined and holomorphic near the contour $|z| = 1$. Then by Proposition 3.4, the contour integral is 0.

Let's see how to obtain the same result using Theorem 3.11 (with $n = 1$, and $f(z) \equiv 1$):

$$\frac{1}{2\pi i} \oint_{|z|=1} \frac{1}{z^{1+1}}\, dz = \left.\frac{d}{dz}\right|_{z=0} 1 = 0.$$

**Example 3.8.** Evaluate the contour integral using higher-order Cauchy's integral formula:

$$\oint_{|z|=1} \frac{e^{2z}}{z^3}\,dz.$$

**Solution**

In practice, it may be helpful to write the higher-order Cauchy's integral formula as:

$$\oint_\gamma \frac{f(z)}{(z-\alpha)^{n+1}}\,dz = \frac{2\pi i}{n!} f^{(n)}(\alpha).$$

Let $f(z) = e^{2z}$ which is entire, then $f'(z) = 2e^{2z}$ and $f''(z) = 4e^{2z}$. By Theorem 3.11 (with $n = 2$), we get:

$$\begin{aligned}\oint_\gamma \frac{e^{2z}}{z^3}\,dz &= \oint_\gamma \frac{e^{2z}}{(z-0)^{2+1}}\,dz \\ &= \frac{2\pi i}{2!} f''(0) = \frac{2\pi i}{2}\cdot 4 = 4\pi i.\end{aligned}$$

**Example 3.9.** Evaluate the contour integral:

$$\oint_{|z|=3} \frac{1}{(z+i)^2(z-2i)^3}\,dz.$$

**Solution**

The contour $|z| = 3$ encloses two singularities of the integrand, namely $-i$ and $2i$. By the hole-drilling technique, we can pick a small $\varepsilon > 0$ such that:

$$\oint_{|z|=3} \frac{1}{(z+i)^2(z-2i)^3}\,dz = \left(\oint_{|z+i|=\varepsilon} + \oint_{|z-2i|=\varepsilon}\right) \frac{1}{(z+i)^2(z-2i)^3}\,dz.$$

Then we calculate each integral on the RHS individually:

$$\begin{aligned}\oint_{|z+i|=\varepsilon} \frac{1}{(z+i)^2(z-2i)^3}\,dz &= \oint_{|z+i|=\varepsilon} \frac{\frac{1}{(z-2i)^3}}{(z+i)^{1+1}}\,dz \\ &= \frac{2\pi i}{1!} \frac{d}{dz}\bigg|_{z=-i} \frac{1}{(z-2i)^3} = -\frac{2\pi i}{3^3}\end{aligned}$$

$$\begin{aligned}\oint_{|z-2i|=\varepsilon} \frac{1}{(z+i)^2(z-2i)^3}\,dz &= \oint_{|z-2i|=\varepsilon} \frac{\frac{1}{(z+i)^2}}{(z-2i)^{2+1}}\,dz \\ &= \frac{2\pi i}{2!} \frac{d^2}{dz^2}\bigg|_{z=2i} \frac{1}{(z+i)^2} \\ &= \pi i \cdot \left[\frac{6}{(z+i)^4}\right]_{z=2i} \\ &= \frac{2\pi i}{3^3}\end{aligned}$$

Therefore,

$$\oint_{|z|=3} \frac{1}{(z+i)^2(z-2i)^3}\,dz = -\frac{2\pi i}{3^3} + \frac{2\pi i}{3^3} = 0.$$

**Exercise 3.19.** Evaluate the following contour integrals:

(a) $\displaystyle\oint_{|z|=2} \frac{\sin z}{(z-\pi)^2}\,dz$

(b) $\displaystyle\oint_{|z|=3} \frac{ze^{tz}}{(z+1)^3}\,dz$ where $t > 0$ is real.

(c) $\displaystyle\oint_{|z|=1} \left(2+z+\frac{1}{z}\right)\frac{f(z)}{z}\,dz$, where $f$ is entire and $f(0) = 1$.

**Exercise 3.20.** Evaluate the contour integral (where $n$ is a positive integer):

$$\oint_{|z|=1} \left(z+\frac{1}{z}\right)^{2n} \frac{1}{z}\,dz.$$

Hence, show that:

$$\int_0^{2\pi} \cos^{2n}\theta\,d\theta = 2\pi\frac{(2n-1)!!}{(2n)!!}.$$

**Exercise 3.21.** Let $f : \Omega \to \mathbb{C}$ be a holomorphic function defined on a simply-connected domain $\Omega$. Suppose $B_R(z_0) \subset \Omega$, show that:

$$\left|f^{(n)}(z_0)\right| \leq \frac{n!}{R^n} \sup_{|z-z_0|=R} |f(z)|$$

for any integer $n \geq 0$.

**3.4.2. Proof of Higher Order Cauchy's Integral Formula.** Now we discuss the proof of Theorem 3.11. From the (zeroth order) Cauchy's integral formula, we know:

$$f(\alpha) = \frac{1}{2\pi i}\oint_\gamma \frac{f(z)}{z-\alpha}\,dz,$$

where $\alpha$ is a point on the domain $\Omega$, and $\gamma$ is a simple closed curve in $\Omega$ enclosing $\alpha$.

Note that if $w \in \mathbb{C}$ is very small, $\alpha + w$ will still be enclosed by $\gamma$, and so we have:

$$f(\alpha+w) = \frac{1}{2\pi i}\oint_\gamma \frac{f(z)}{(z-\alpha-w)}\,dz.$$

Our first goal is to show Theorem 3.11 holds for $f'(\alpha)$, i.e. $n = 1$. Recall that:

$$f'(\alpha) = \lim_{w\to 0}\frac{f(\alpha+w)-f(\alpha)}{w}.$$

We will use the zeroth order Cauchy's integral formula to evaluate such a limit:

$$\begin{aligned} f'(\alpha) &= \frac{1}{2\pi i}\lim_{w\to 0}\frac{1}{w}\left(\oint_\gamma \frac{f(z)}{z-\alpha-w}\,dz - \oint_\gamma \frac{f(z)}{z-\alpha}\right)dz \\ &= \frac{1}{2\pi i}\lim_{w\to 0}\oint_\gamma f(z)\cdot\frac{1}{w}\left(\frac{1}{z-\alpha-w}-\frac{1}{z-\alpha}\right)dz. \end{aligned} \tag{3.6}$$

By straight-forward computation, we get:

$$\frac{1}{w}\left(\frac{1}{z-\alpha-w}-\frac{1}{z-\alpha}\right) = \frac{1}{(z-\alpha-w)(z-\alpha)}.$$

The integrand of (3.6) becomes $\dfrac{f(z)}{(z-\alpha-w)(z-\alpha)}$, which is bounded as $z \in \gamma$ is away from $\alpha$ and $\alpha + w$ when $w$ is small, and that the holomorphic function $f$ is bounded on $\gamma$ by Extreme-Value Theorem. The length of $\gamma$ is also bounded. Using Lebesgue

Dominated Covergence Theorem (commonly called LDCT in short), we can switch the limit and the integral sign of (3.6), and get:

$$\begin{aligned} f'(\alpha) &= \frac{1}{2\pi i}\oint_\gamma \lim_{w\to 0} f(z)\cdot\frac{1}{w}\left(\frac{1}{z-\alpha-w}-\frac{1}{z-\alpha}\right)dz \\ &= \frac{1}{2\pi i}\oint_\gamma \lim_{w\to 0}\frac{f(z)}{(z-\alpha-w)(z-\alpha)}\,dz \\ &= \frac{1}{2\pi i}\oint_\gamma \frac{f(z)}{(z-\alpha)^2}\,dz, \end{aligned}$$

proving Theorem 3.11 when $n = 1$.

The second and higher order cases of Theorem 3.11 can be proved by induction. Assume the theorem holds for some integer $n$:

$$f^{(n)}(\alpha) = \frac{n!}{2\pi i}\oint_\gamma \frac{f(z)}{(z-\alpha)^{n+1}}\,dz$$

for any $\alpha$ enclosed by $\gamma$. When $w$ is very small, $\alpha + w$ is also enclosed by $\gamma$, hence it is also true that:

$$f^{(n)}(\alpha+w) = \frac{n!}{2\pi i}\oint_\gamma \frac{f(z)}{(z-\alpha-w)^{n+1}}\,dz.$$

Our next goal is to determine $f^{(n+1)}(\alpha)$ from the definition:

$$f^{(n+1)}(\alpha) = \lim_{w\to 0}\frac{f^{(n)}(\alpha+w)-f^{(n)}(\alpha)}{w}.$$

We leave it as an exercise:

**Exercise 3.22.** Follow the outline listed below, and complete the inductive proof of Theorem 3.11:

(a) Show that:

$$\begin{aligned} &\frac{1}{w}\left(\frac{1}{(z-\alpha-w)^{n+1}}-\frac{1}{(z-\alpha)^{n+1}}\right) \\ &= \frac{1}{(z-\alpha-w)(z-\alpha)}\sum_{j=0}^{n}\frac{1}{(z-\alpha-w)^j(z-\alpha)^{n-j}} \end{aligned}$$

(b) Using the induction assumption and LDCT, show that

$$\begin{aligned} &f^{(n+1)}(\alpha) \\ &= \frac{n!}{2\pi i}\oint_\gamma \lim_{w\to 0}\frac{f(z)}{(z-\alpha-w)(z-\alpha)}\sum_{j=0}^{n}\frac{1}{(z-\alpha-w)^j(z-\alpha)^{n-j}}\,dz. \end{aligned}$$

(c) Finally, complete the proof.

**3.4.3. Liouville's Theorem.** We now discuss an important consequence (Liouville's Theorem) of the higher order Cauchy's integral formula. Using this theorem, one can give a very short and elegant proof that every non-constant complex polynomial must have at least one root!

**Theorem 3.13** (Liouville's Theorem)**.** *Any bounded entire function must be constant.*

**Proof.** The proof is a consequence of 1st-order Cauchy's integral formula. Suppose $f : \mathbb{C} \to \mathbb{C}$ is a entire function and that there exists $M > 0$ such that $|f(z)| \leq M$ for any $z \in \mathbb{C}$.

Take an arbitrary $\alpha \in \mathbb{C}$, and consider the contour $|z-\alpha| = R$. By Theorem 3.11 with $n = 1$, we know:

$$f'(\alpha) = \frac{1}{2\pi i} \oint_{|z-\alpha|=R} \frac{f(z)}{(z-\alpha)^2}\, dz.$$

Then on the contour, we have:

$$\left| \frac{f(z)}{(z-\alpha)^2} \right| \le \frac{M}{R^2},$$

and by Lemma 3.6, we can estimate that:

$$\left| \oint_{|z-\alpha|=R} \frac{f(z)}{(z-\alpha)^2}\, dz \right| \le 2\pi R \cdot \frac{M}{R^2} = \frac{2\pi M}{R}.$$

Therefore, we have for any $\alpha \in \mathbb{C}$ and $R > 0$:

$$|f'(\alpha)| = \left| \frac{1}{2\pi i} \oint_{|z-\alpha|=R} \frac{f(z)}{(z-\alpha)^2}\, dz \right| \le \frac{1}{2\pi} \cdot \frac{2\pi M}{R} \to 0 \text{ as } R \to +\infty.$$

This shows $f' \equiv 0$, and hence $f$ is a constant function. □

**Exercise 3.23.** Why is it necessary that $f$ is entire in the proof of Liouville's Theorem? Which step will it break down if $f$ is holomorphic only on a proper subset of $\mathbb{C}$?

**Exercise 3.24.** Prove the following general version of Liouville's Theorem: Suppose $f : \mathbb{C} \to \mathbb{C}$ is an entire function, and there exists $M > 0$ and a nonnegative integer $k$ such that:

$$|f(z)| \le M\,|z|^k \quad \text{for any } z \in \mathbb{C}.$$

Show that $f$ is a polynomial of degree at most $k$.

**Exercise 3.25.** Suppose $f : \mathbb{C} \to \mathbb{C}$ is an entire function satisfying:

$$\lim_{R\to+\infty} \sup_{|z|\ge R} \frac{|f(z)|}{R} = 0.$$

Show that $f$ is a constant function.

Liouville's Theorem is a "luxury" for holomorphic functions. There are many non-constant bounded functions $f : \mathbb{R} \to \mathbb{R}$ that are (real) differentiable everywhere, while Liouville's Theorem says there is no non-constant bounded functions $f : \mathbb{C} \to \mathbb{C}$ which are complex differentiable everywhere.

The theorem has many surprising consequences. One of them is:

**Corollary 3.14** (Fundamental Theorem of Algebra)**.** *Every non-constant complex polynomial* $p(z) = a_n z^n + a_{n-1} z^{n-1} + \ldots + a_1 z + a_0$ *must have at least one complex root.*

**Proof.** We prove by contradiction. If $p(z)$ has no root, then $\frac{1}{p(z)}$ is an entire function. Note that $|p(z)| \to \infty$ as $|z| \to \infty$, we have: $\frac{1}{p(z)} \to 0$ as $|z| \to \infty$. In particular, $\frac{1}{p(z)}$ is bounded. By Liouville's Theorem, $\frac{1}{p(z)}$ is constant, which is a contradiction. □

**Remark 3.15.** There are many proofs of Fundamental Theorem of Algebra, at least one in almost all important fields in mathematics. There is one in Topology using the concept of homotopy. There is even one geometric proof using Gauss-Bonnet's Theorem in Differential Geometry! Ironically, despite the name of the theorem, a purely algebraic proof has not yet been found. The most purest algebraic proof uses Galois Theory, but that proof is based on the fact that every real number has a real cubic root (which has to be justified using Intermediate-Value Theorem in Real Analysis).

**Exercise 3.26.** In the proof of Fundamental Theorem of Algebra (Corollary 3.14), we used the fact that $|p(z)| \to \infty$ as $|z| \to \infty$. Although this fact is intuitively clear since the dominant term $a_n z^n$ of $p$ becomes very large when $|z| \to \infty$, try to prove this fact in a more rigorous way. Hint: try to show that if $p(z) = a_n z^n + a_{n-1} z^{n-1} + \ldots + a_0$, then

$$|p(z)| \geq |z|^{n-1} \left(|a_n z| - |a_{n-1}| - \ldots - |a_0|\right)$$

whenever $|z| > 1$.

**Exercise 3.27.** Using Liouville's Theorem, show that if the image of an entire function $f : \mathbb{C} \to \mathbb{C}$ is disjoint from an open ball $B_\delta(z_0)$, then $f$ is a constant function.

The above exercise gives a very powerful way for showing certain entire function must be constant. For example, if $f : \mathbb{C} \to \mathbb{C}$ is entire and maps $\mathbb{C}$ onto the upper-half plane in $\mathbb{C}$, then the image of $f$ is disjoint from many open balls such as $B_{1/2}(-i)$. Hence it must be a constant.

## 3.5. Morera's Theorem

Before we stated Morera's Theorem, let's recall the proof of Cauchy-Goursat's Theorem. Using the holomorphic condition on $f$, Step 1 shows that $\oint_T f(z)\,dz = 0$ for any triangle contour $T$ in the domain. Using this fact, Step 2 shows $F(z) := \int_{L(z_0,z)} f(\xi)\,d\xi$, where $L(z_0, z)$ is the straight path from a fixed point $z_0$ to $z$, is a primitive function for $f$, i.e. $F'(z) = f(z)$ on the convex domain $\Omega$.

It is a nice observation that the proof in Step 2 requires only two facts about $f$, namely:

(1) $f$ is continuous on $\Omega$; and

(2) $\oint_T f(z)\,dz = 0$ for any triangle $T$ in $\Omega$.

Under these two conditions, the entire argument in Step 2 is still valid even if we don't assume that $f$ is holomorphic on $\Omega$. Step 2 shows $F'(z) = f(z)$ on $\Omega$, hence proving $\oint_\gamma f(z)\,dz = 0$ for any closed curve $\gamma$ in $\Omega$.

The result that $F'(z) = f(z)$ on $\Omega$ has another implication: since the primitive function $F$ is holomorphic on $\Omega$ (and its derivative is $f$), the higher order Cauchy's integral formula (Theorem 3.11) and Corollary 3.12 tell us that $F$ is complex differentiable on $\Omega$ for infinitely many times. Certainly, it shows $f = F'$ is also complex differentiable on $\Omega$ for infinitely many times too. In particular, $f$ is holomorphic on $\Omega$.

To summarize, the preceding discussion proves the following remarkable result:

**Theorem 3.16** (Morera's Theorem). *If $f : \Omega \to \mathbb{C}$ is a continuous function on an open domain $\Omega$, and*
$$\oint_T f(z)\,dz = 0$$
*for any triangle contour $T$ in $\Omega$, then $f$ is holomorphic on $\Omega$.*

**Remark 3.17.** Although convexity of the domain is needed in Step 2 of the proof of Cauchy-Goursat's Theorem, we do not need to assume $\Omega$ is convex when using Morera's Theorem. It is because complex differentiability is a *local* property. One can first restrict $f$ on an open ball $B_\varepsilon(z_0)$ which is convex, then prove $f$ is holomorphic on $B_\varepsilon(z_0)$. Simply repeat the same argument on all other open balls in the domain. It will show $f$ is holomorphic on the whole $\Omega$.

In practice, it seems more difficult to verify $\oint_T f(z)\,dz = 0$ for any triangle $T$ than to show $f$ is holomorphic directly. Nonetheless, Morera's Theorem can come in handy if we want to show holomorphicity of a function which is not quite explicit. In the last chapter, we may encounter functions defined in an integral form, such as the Gamma's function:
$$\Gamma(z) = \int_0^\infty t^{z-1}e^{-t}\,dt.$$
It is almost impossible to find an explicit, integral-free expression. Nonetheless, it is possible to show it is a holomorphic function using Morera's Theorem. The key idea is to show that $\oint_T \Gamma(z)\,dz = 0$ for any triangle $T$ in the domain under consideration.

**Example 3.10.** Define $f : \Omega := \{z : \text{Re}(z) < 0\} \to \mathbb{C}$ by:

$$f(z) = \int_0^\infty \frac{e^{zt}}{t+1}\, dt.$$

Show that $f(z)$ is holomorphic on $\Omega$.

**Solution**

First we show that $f$ is defined on $\Omega$: for any $z \in \Omega$ and $t \in [0, \infty)$, we have:

$$\left| \frac{e^{zt}}{t+1} \right| \leq |e^{zt}| \leq e^{xt}$$

(as usual, we denote $z = x + yi$). Note that:

$$\int_0^\infty e^{xt}\, dt = \left[ \frac{1}{x} e^{xt} \right]_0^\infty = -\frac{1}{x} < \infty.$$

Hence, $\displaystyle\int_0^\infty \frac{e^{zt}}{t+1}\, dt$ is integrable.

It is quite difficult to find an explicit formula for $f(z)$, let alone its derivative. To show it is holomorphic, we are going to use Morera's Theorem: given any triangle $T$ in $\Omega$, we want to show $\displaystyle\int_T f(z)\, dz = 0$.

$$\begin{aligned}
\int_T f(z)\, dz &= \int_T \int_0^\infty \frac{e^{zt}}{t+1}\, dt\, dz \\
&= \int_0^\infty \int_T \frac{e^{zt}}{t+1}\, dz\, dt && \text{(Fubini's Theorem)} \\
&= \int_0^\infty 0\, dt && \left(\text{since } \frac{e^{zt}}{t+1} \text{ is holomorphic}\right) \\
&= 0
\end{aligned}$$

To justify the legitimacy of using Fubini's Theorem, we require the integral $\displaystyle\int_T \int_0^\infty \left| \frac{e^{zt}}{t+1} \right| dt\, |dz|$ to be finite. To verify this, we consider $\displaystyle\int_0^\infty \left| \frac{e^{zt}}{t+1} \right| dt \leq -\frac{1}{x}$, so that $\displaystyle\int_T \int_0^\infty \left| \frac{e^{zt}}{t+1} \right| dt\, |dz| \leq \int_T -\frac{1}{x}\, |dz|$, which is finite since $x$ is away from 0 when $z$ is on any triangle $T \subset \Omega$.

Hence by Morera's Theorem, $f$ is holomorphic on $\Omega$.

**Exercise 3.28.** Define $f : \mathbb{C} \backslash [0, 1] \to \mathbb{C}$ by:

$$f(z) = \int_0^1 \frac{\sqrt{t}}{t - z}\, dt.$$

Show that $f$ is holomorphic on its domain.

**Exercise 3.29.** Suppose $\{f_n\}_{n=1}^\infty$ is a sequence of holomorphic functions on an open domain $\Omega$, and that $f_n$ converges uniformly to $f$ on $\Omega$. Show that the limit function $f$ is also holomorphic on $\Omega$.

**Exercise 3.30.** Recall that the Riemann's zeta function $\zeta : \Omega \to \mathbb{C}$ is defined on $\Omega := \{z : \mathrm{Re}(z) > 1\}$ and by:

$$\zeta(z) := \sum_{n=1}^{\infty} \frac{1}{n^z} = \sum_{n=1}^{\infty} \frac{1}{e^{z \ln n}}.$$

(a) Show that the series $\sum_{n=1}^{\infty} \frac{1}{n^z}$ converges uniformly on $\Omega_\varepsilon := \{z : \mathrm{Re}(z) > 1 + \varepsilon\}$ for any $\varepsilon > 0$.

(b) Show that $\zeta$ is holomorphic on $\Omega$.