Chapter 1

Preliminaries

1.1. Complex Numbers

1.1.1. Basic Arithmetics. From middle/high school, we learned that the quadratic equation $x^2 + 1 = 0$ does not have any real root because $x^2 + 1 > 0$ for any $x \in \mathbb{R}$. Complex numbers are introduced to make it possible for the equation $x^2 + 1 = 0$ to have roots. We denote:

$$i = \sqrt{-1} \quad \text{so that} \quad i^2 = -1.$$

While complex numbers make their appearance for purely algebraic purposes, their uses branch out to many scientific fields beyond Mathematics, including Quantum Mechanics, String Theory, Electrical Engineering, Fluid Mechanics, etc.

Definition 1.1 (Complex Numbers)**.** A *complex number* z is a number of the form:

$$z = a + bi$$

where a and b are real numbers, and $i = \sqrt{-1}$. We call:

- a is the *real part* of z and is denoted by $a =: \mathrm{Re}(z)$; and
- b is the *imaginary part* of z and is denoted by $b =: \mathrm{Im}(z)$.

The set of all complex numbers is denoted by $\mathbb{C}$. Precisely, we have:

$$\mathbb{C} := \{a + bi : a, b \in \mathbb{R}\}.$$

Remark 1.2. Note that a real number is also considered as a complex number, since $a = a + 0i$. In other words, we have $\mathbb{R} \subset \mathbb{C}$.

A complex number $z = x + yi$ can be geometrically represented by the point (x, y) in $\mathbb{R}^2$ (see Figure 1.1). The x-axis is now called the *real axis* as it represents numbers of the form $a + 0i$. Likewise, the y-axis is called the *imaginary axis*, which represents numbers of the form $0 + bi$.

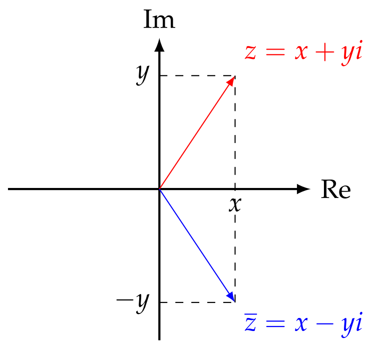


Figure 1.1. geometry of complex numbers

Given two complex numbers $z_1 = a + bi$ and $z_2 = c + di$, the arithmetics between them are defined by:

$$\begin{aligned} z_1 + z_2 &= (a + c) + (b + d)i \\ z_1 - z_2 &= (a - c) + (b - d)i \\ z_1 z_2 &= (a + bi)(c + di) \\ &= (ac - bd) + (ad + bc)i \\ \frac{z_1}{z_2} &= \frac{a + bi}{c + di} \cdot \frac{c - di}{c - di} && \text{(where } z_2 \neq 0\text{)} \\ &= \frac{(ac + bd)}{c^2 + d^2} + \frac{(bc - ad)i}{c^2 + d^2} \end{aligned}$$

1.1.2. Conjugate and Modulus. Two important operations on complex numbers are taking *conjugates* and *modulus*:

Definition 1.3 (Conjugate and Modulus)**.** Given $z = a + bi \in \mathbb{C}$, we denote and define:

- $\overline{z} := a - bi$ as the *conjugate* of z; and
- $|z| := \sqrt{a^2 + b^2}$ as the *modulus* of z.

Remark 1.4. It is important to note that complex numbers are *un-ordered*. It does not make sense to say $z_1 < z_2$ or $z_1 > z_2$. However, since $|z|$ is a real number, it makes sense to make comparison of $|z_1|$ and $|z_2|$.

Remark 1.5. Geometrically, $\overline{z}$ is obtained by reflecting z across the Re-axis (see Figure 1.1), and $|z|$ is the magnitude of the position vector representing z.

Listed below are some very useful properties of complex numbers. Given any $z, z_1, z_2 \in \mathbb{C}$, we have:

$$z\overline{z} = |z|^2 \qquad \overline{\overline{z}} = z \qquad |\overline{z}| = |z|$$

$$\mathrm{Re}(z) = \frac{z + \overline{z}}{2} \qquad \mathrm{Im}(z) = \frac{z - \overline{z}}{2i}$$

$$\overline{z_1 \pm z_2} = \overline{z_1} \pm \overline{z_2} \qquad \overline{z_1 z_2} = \overline{z_1}\,\overline{z_2} \qquad \overline{\left(\frac{z_1}{z_2}\right)} = \frac{\overline{z_1}}{\overline{z_2}}$$

The proofs are all straight-forward and hence omitted. Simply let $z = x + yi$ and verify LHS and RHS are equal in each property. Let's look at some examples on how to make good use of these properties:

Example 1.1. Show that for any $z_1, z_2 \in \mathbb{C}$, we have:

$$|z_1 + z_2|^2 = |z_1|^2 + |z_2|^2 + 2\text{Re}(z_1\overline{z_2}).$$

Solution

The key step is to use the property that $|z|^2 = z\overline{z}$ for any $z \in \mathbb{C}$.

$$\begin{aligned}
|z_1 + z_2|^2 &= (z_1 + z_2)\overline{(z_1 + z_2)} = (z_1 + z_2)(\overline{z_1} + \overline{z_2}) \\
&= z_1\overline{z_1} + z_1\overline{z_2} + \overline{z_1}z_2 + z_2\overline{z_2} \\
&= |z_1|^2 + z_1\overline{z_2} + \overline{z_1\overline{z_2}} + |z_2|^2 \\
&= |z_1|^2 + |z_2|^2 + 2\text{Re}(z_1\overline{z_2})
\end{aligned}$$

Example 1.2. Let $\alpha, \beta \in \mathbb{C}\backslash\{0\}$. Show that $\alpha\overline{\beta} \in \mathbb{R}$ if and only if $\frac{\beta}{\alpha} \in \mathbb{R}$.

Solution

($\Longrightarrow$) Suppose $\alpha\overline{\beta} \in \mathbb{R}$, then we have $\overline{\alpha\overline{\beta}} = \alpha\overline{\beta}$, and so $\overline{\alpha}\beta = \alpha\overline{\beta}$. Since $\alpha, \beta \neq 0$, by rearrangement we get:

$$\frac{\beta}{\alpha} = \frac{\overline{\beta}}{\overline{\alpha}} = \overline{\left(\frac{\beta}{\alpha}\right)}$$

Therefore, $\frac{\beta}{\alpha}$ is equal to its conjugate. It concludes that $\frac{\beta}{\alpha} \in \mathbb{R}$.

($\Longleftarrow$) Conversely, let $\frac{\beta}{\alpha} = \lambda \in \mathbb{R}$. Then: $\alpha\overline{\beta} = \alpha\overline{\lambda\alpha} = \lambda\alpha\overline{\alpha} = \lambda\,|\alpha|^2 \in \mathbb{R}$.

It is important to note that in general $|z_1 + z_2| \neq |z_1| + |z_2|$. However, we do have:

Proposition 1.6 (Triangle Inequality). *Let $z_1, z_2 \in \mathbb{C}$, we have:*

$$|z_1 + z_2| \leq |z_1| + |z_2|\,.$$

Proof. From Example 1.1, we have:

$$|z_1 + z_2|^2 = |z_1|^2 + |z_2|^2 + 2\text{Re}(z_1\overline{z_2}).$$

Let $z_1\overline{z_2} = u + vi$, where $u, v \in \mathbb{R}$. Then, we have:

$$2\text{Re}(z_1\overline{z_2}) = 2u \leq 2\sqrt{u^2 + v^2} = 2\,|z_1\overline{z_2}| = 2\,|z_1|\,|\overline{z_2}| = 2\,|z_1|\,|z_2|\,.$$

Finally, we get:

$$|z_1 + z_2|^2 \leq |z_1|^2 + |z_2|^2 + 2\,|z_1|\,|z_2| = (|z_1| + |z_2|)^2$$

and it completes the proof by taking square root on both sides. □

Exercise 1.1. Let $z_1, z_2 \in \mathbb{C}$, show that:

$$|z_1 + z_2|^2 + |z_1 - z_2|^2 = 2(|z_1|^2 + |z_2|^2).$$

Exercise 1.2. Let $\alpha, \beta \in \mathbb{C}$. Suppose $\alpha\overline{z} + \beta z \in \mathbb{R}$ for *any* $z \in \mathbb{C}$. Show that $\alpha = \overline{\beta}$.

Exercise 1.3. Let $z_1, z_2 \in \mathbb{C}$. Show that $||z_1| - |z_2|| \leq |z_1 - z_2|$.

Exercise 1.4. Let p be the polynomial $p(z) = c_0 + c_1 z + \cdots + c_d z^d$ where $d \geq 1$ and $\{c_0, c_1, c_2, \ldots, c_d\}$ is a (monotone) decreasing sequence of positive real numbers. Prove that the polynomial equation $p(z) = 0$ does not have any roots with modulus (strictly) less than 1.

1.1.3. Polar Form. There are two common types of coordinates in $\mathbb{R}^2$, namely rectangular and polar. Apart from the standard (rectangular) form $x + yi$ for representing a complex number, we can also represent a complex number by a polar form. The conversion rule between rectangular and polar coordaintes is given by:

$$x = r\cos\theta$$
$$y = r\sin\theta$$

Therefore, a complex number $z = x + yi$ can be written as:

$$z = (r\cos\theta) + i(r\sin\theta) = r(\cos\theta + i\sin\theta).$$

The form $z = r(\cos\theta + i\sin\theta)$ is commonly called the *polar form* of z.

Note that $|\cos\theta + i\sin\theta| = \sqrt{\cos^2\theta + \sin^2\theta} = 1$. When $z = r(\cos\theta + i\sin\theta)$, it is easy to see that $r = |z|$. However, the value of θ is not unique as both sin and cos are periodic functions of period 2π. We define the principal argument of a complex number to be the angle θ with a specified range described below:

Definition 1.7 (Principal Argument)**.** Given a complex number z, the *principal argument* of z, denoted by $\operatorname{Arg}(z)$, is defined to be the angle $\theta_0 \in (-\pi, \pi]$ such that:

$$z = |z|\,(\cos\theta_0 + i\sin\theta_0).$$

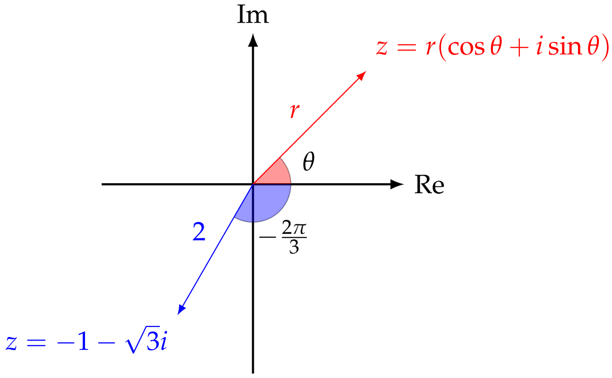
For example, $-1 - \sqrt{3}i$ has modulus 2 and so the r-coordinate is 2:

$$-1 - \sqrt{3}i = 2\left(-\frac{1}{2} - \frac{\sqrt{3}}{2}i\right).$$

To find the θ-coordinate, we solve $\cos\theta = -\frac{1}{2}$ and $\sin\theta = -\frac{\sqrt{3}}{2}$. From standard trigonometry, we get $\theta = \frac{4\pi}{3} + 2k\pi$ for any integer k. The only θ that falls into the range $(-\pi, \pi]$ is $-\frac{2\pi}{3} = \frac{4\pi}{3} - 2\pi$. Therefore, we have:

$$-1 - \sqrt{3}i = 2\left(\cos\left(-\frac{2\pi}{3}\right) + i\sin\left(-\frac{2\pi}{3}\right)\right)$$

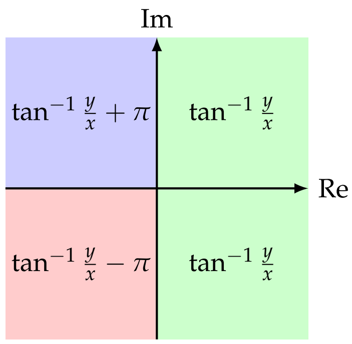
and $\operatorname{Arg}(-1 - \sqrt{3}i) = -\frac{2\pi}{3}$.



In general, $\operatorname{Arg}(x+yi)$ can be found using $\tan^{-1}\frac{y}{x}$ since if $x = r\cos\theta$ and $y = r\sin\theta$, then $\tan\theta = \frac{y}{x}$. However, it is important to note that $\operatorname{Arg}(x+yi)$ is NOT simply equal to $\tan^{-1}\frac{y}{x}$ because by definition of the inverse tangent, $\tan^{-1}\frac{y}{x}$ takes the value in $(-\frac{\pi}{2}, \frac{\pi}{2})$ only. Precisely, we have (when $x \neq 0$):

$$\operatorname{Arg}(x+yi) = \begin{cases} \tan^{-1}\frac{y}{x} & \text{if } (x,y) \text{ is in 1st and 4th quadrants;} \\ \tan^{-1}\frac{y}{x} + \pi & \text{if } (x,y) \text{ is in 2nd quadrant;} \\ \tan^{-1}\frac{y}{x} - \pi & \text{if } (x,y) \text{ is in 3rd quadrant;} \end{cases}$$

Furthermore, $\operatorname{Arg}(0+yi) = \frac{\pi}{2}$ when $y > 0$; and $\operatorname{Arg}(0+yi) = -\frac{\pi}{2}$ when $y < 0$. Note that $\operatorname{Arg}(0+0i)$ is undefined.



Exercise 1.5. Express the following complex numbers in polar form, and find their principal arguments Arg:

(a) $1 + 2i$

(b) $1 - 2i$

(c) $\cos(-\pi) + i\sin(-\pi)$

(d) $-i$

Exercise 1.6. Given $|z| = 1$, show that:

(a) $\operatorname{Re}\left(\frac{1+z}{1-z}\right) = 0$

(b) $\left|\frac{z-\omega}{1-\overline{\omega}z}\right| = 1$ for any $\omega \in \mathbb{C}$ such that $\overline{\omega}z \neq 1$.

Exercise 1.7. Given $z, \omega \in \mathbb{C}$ such that $|z+\omega| = |z-\omega|$, show that:

(a) $iz\overline{\omega} \in \mathbb{R}$

(b) $\operatorname{Arg}(z) - \operatorname{Arg}(\omega) = \frac{\pi}{2}$ or $\frac{3\pi}{2}$

Exercise 1.8. Show that the real-valued function $f : \mathbb{R}^2 \backslash \{(x,0) : x \leq 0\}$ defined by $f(x,y) := \operatorname{Arg}(x+yi)$ is continuous.

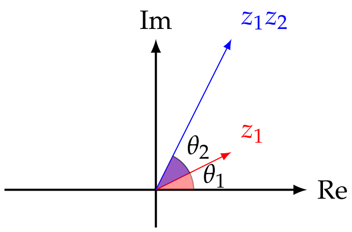
1.1.4. De Moivre's Theorem. By expressing complex numbers using polar form, one can see that multiplications and divisions between two complex numbers are *rotations* in the complex plane $\mathbb{C}$. It thanks to the fact that:

$$\begin{aligned}(1.1) \qquad &(\cos\theta + i\sin\theta)(\cos\phi + i\sin\phi)\\ &= (\cos\theta\,\cos\phi - \sin\theta\,\sin\phi) + i(\cos\theta\,\sin\phi + \sin\theta\,\cos\phi)\\ &= \cos(\theta+\phi) + i\sin(\theta+\phi)\end{aligned}$$

Using (1.1), we can see that given $z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$ and $z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$, then we have:

$$\begin{aligned}z_1z_2 &= r_1r_2(\cos\theta_1 + i\sin\theta_1)(\cos\theta_2 + i\sin\theta_2)\\ &= r_1r_2(\cos(\theta_1+\theta_2) + i\sin(\theta_1+\theta_2))\end{aligned}$$

Therefore, z_1z_2 is obtained by rotating z_1 by $\operatorname{Arg}(z_2)$, and lengthen (or shorten) z_1 by a factor of $|z_2|$. See the figure below:



An important consequence of (1.1) is the following celebrated theorem:

Theorem 1.8 (De Moivre's Theorem). *For any $\theta \in \mathbb{R}$ and $n \in \mathbb{Z}$, we have:*

$$(1.2) \qquad (\cos\theta + i\sin\theta)^n = \cos(n\theta) + i\sin(n\theta).$$

Proof. We prove by induction for positive n's. Clearly (1.2) is true when $n = 1$. Assume that (1.2) is true when $n = k$ for some positive integer k. Then, for $n = k+1$, we have:

$$\begin{aligned}(\cos\theta + i\sin\theta)^{k+1} &= (\cos\theta + i\sin\theta)^k(\cos\theta + i\sin\theta)\\ &= (\cos(k\theta) + i\sin(k\theta))(\cos\theta + i\sin\theta) \quad \text{(induction assumption)}\\ &= \cos(k\theta+\theta) + i\sin(k\theta+\theta) \quad \text{(from (1.1))}\\ &= \cos((k+1)\theta) + i\sin((k+1)\theta)\end{aligned}$$

Hence (1.2) is true when $n = k+1$. By induction, (1.2) is true for all positive integer n.

When $n = 0$, (1.2) also holds because $(\cos\theta + i\sin\theta)^0 = 1$.

Finally we consider negative integers n. When $n < 0$, let $m = -n$ then m is a positive integer. From above, (1.2) holds for this m:

$$\begin{aligned}
(\cos\theta + i\sin\theta)^m &= \cos(m\theta) + i\sin(m\theta)\\
(\cos\theta + i\sin\theta)^{-n} &= \cos(-n\theta) + i\sin(-n\theta)\\
\frac{1}{(\cos\theta + i\sin\theta)^n} &= \cos(n\theta) - i\sin(n\theta)\\
(\cos\theta + i\sin\theta)^n &= \frac{1}{\cos(n\theta) - i\sin(n\theta)}\\
&= \frac{1}{\cos(n\theta) - i\sin(n\theta)} \cdot \frac{\cos(n\theta) + i\sin(n\theta)}{\cos(n\theta) + i\sin(n\theta)}\\
&= \frac{\cos(n\theta) + i\sin(n\theta)}{\cos^2(n\theta) + \sin^2(n\theta)} = \cos(n\theta) + i\sin(n\theta).
\end{aligned}$$

This proves (1.2) holds for negative integers n, and hence completing the proof of the theorem. □

De Moivre's Theorem can be used to derive some trigonometric identities. For example, consider $(\cos\theta + i\sin\theta)^3$. On one hand, De Moivre's Theorem shows that:

$$(\cos\theta + i\sin\theta)^3 = \cos 3\theta + i\sin 3\theta$$

and on the other hand, by expanding $(\cos\theta + i\sin\theta)^3$ we get:

$$\begin{aligned}
(\cos\theta + i\sin\theta)^3 &= \cos^3\theta + 3(\cos^2\theta)(i\sin\theta) + 3\cos\theta(i\sin\theta)^2 + (i\sin\theta)^3\\
\cos 3\theta + i\sin 3\theta &= (\cos^3\theta - 3\cos\theta\sin^2\theta) + i(3\cos^2\theta\sin\theta - \sin^3\theta)
\end{aligned}$$

By equating the real and imaginary parts, we get:

$$\begin{aligned}
\cos 3\theta &= \cos^3\theta - 3\cos\theta\sin^2\theta = \cos^3\theta - 3\cos\theta\,(1-\cos^2\theta) = 4\cos^3\theta - 3\cos\theta\\
\sin 3\theta &= 3\cos^2\theta\sin\theta - \sin^3\theta = 3(1-\sin^2\theta)\sin\theta - \sin^3\theta = 3\sin\theta - 4\sin^3\theta
\end{aligned}$$

Exercise 1.9. Use De Moivre's Theorem to show that:

$$\cos n\theta = \sum_{k=0}^{[\frac{n}{2}]} \sum_{r=0}^{k} C_{2k}^n C_r^k (-1)^{k+r} \cos^{n-2k+2r}\theta$$

for any $n \in \mathbb{N}$. Here $[\frac{n}{2}]$ denotes the integer part of $\frac{n}{2}$.

1.1.5. Roots of Complex Numbers. In the real number system, the root equation $x^n = a$ where $a \neq 0$ and $n \in \mathbb{N}$, has at most two solutions. When n is odd (no matter whether a is positive or negative), the only real solution is $x = \sqrt[n]{a}$. When n is even and $a > 0$, there are two real solutions $x = \sqrt[n]{a}$ or $-\sqrt[n]{a}$. The equation has no solution when n is even and $a < 0$.

However, in the complex number system, the root equation $z^n = a$, where $a \in \mathbb{C}\backslash\{0\}$ and $n \in \mathbb{N}$, always has n solutions! Let's first look at the simplest equation $z^n = 1$:

Certainly, 1 is a solution to the equation. Furthermore, using De Moivre's Theorem, we get:

$$\left(\cos\frac{2\pi}{n} + i\sin\frac{2\pi}{n}\right)^n = \cos\left(\frac{2\pi}{n}\cdot n\right) + i\sin\left(\frac{2\pi}{n}\cdot n\right) = \cos(2\pi) + i\sin(2\pi) = 1.$$

Clearly, this shows the complex number $\cos\frac{2\pi}{n} + i\sin\frac{2\pi}{n}$ satisfies the equation $z^n = 1$. In fact, any number which can be expressed in form of $\cos\frac{2k\pi}{n} + i\sin\frac{2k\pi}{n}$, where k is an integer, is a solution to the root equation $z^n = 1$:

$$\left(\cos\frac{2k\pi}{n} + i\sin\frac{2k\pi}{n}\right)^n = \cos(2k\pi) + i\sin(2k\pi) = 1.$$

Note that the set of roots $\left\{\cos\frac{2k\pi}{n} + i\sin\frac{2k\pi}{n} : k \in \mathbb{Z}\right\}$ consists of n distinct elements only (instead of infinitely many), since

$$\cos\frac{2k\pi}{n} + i\sin\frac{2k\pi}{n} = \cos\frac{2m\pi}{n} + i\sin\frac{2m\pi}{n}$$

if and only if $k - m$ is a multiple of n. In other words, when $k = n$, the root $\cos\frac{2k\pi}{n} + i\sin\frac{2k\pi}{n}$ is the same as the one with $k = 0$. Likewise, the root when $k = n + 1$ gives the same root as the one with $k = 1$, etc. Overall, the set of n-th roots of 1 is essentially given by the finite set:

$$\left\{\cos\frac{2k\pi}{n} + i\sin\frac{2k\pi}{n} : k \in \{0, 1, 2, \ldots, n-1\}\right\}$$

and these n numbers are called the *n-th root of* 1. In terms of notations, we write:

$$1^{\frac{1}{n}} = \left\{\cos\frac{2k\pi}{n} + i\sin\frac{2k\pi}{n} : k \in \{0, 1, 2, \ldots, n-1\}\right\}.$$

It is important to note that unlike the real number system, the n-th root of 1 is no longer a single number. In contrast, $1^{\frac{1}{n}}$ represents a *set* of roots for the equation $z^n = 1$.

Due to this distinctive difference from the real number system, from now on we will use $\sqrt[n]{a}$ to denote the n-th root of a in the *real* number system; while we will use $a^{\frac{1}{n}}$ to denote the n-th root of a in the *complex* number system, which will be discussed in the next paragraph.

Now consider the general root equation $z^n = a$ where $a \neq 0$. Suppose a can be expressed in polar form as:

$$a = |a|(\cos\theta + i\sin\theta)$$

Then, one can show that:

$$\underbrace{\sqrt[n]{|a|}}_{\text{real } n\text{-th root}} \left(\cos\left(\frac{\theta + 2k\pi}{n}\right) + i\sin\left(\frac{\theta + 2k\pi}{n}\right)\right), \qquad k \in \mathbb{Z}$$

are solutions to the root equation $z^n = a$, since:

$$\begin{aligned}
&\left[\sqrt[n]{|a|}\left(\cos\left(\frac{\theta + 2k\pi}{n}\right) + i\sin\left(\frac{\theta + 2k\pi}{n}\right)\right)\right]^n \\
&= \left(\sqrt[n]{|a|}\right)^n \left(\cos\left(\frac{\theta + 2k\pi}{n} \cdot n\right) + i\sin\left(\frac{\theta + 2k\pi}{n} \cdot n\right)\right) \\
&= |a|(\cos(\theta + 2k\pi) + i\sin(\theta + 2k\pi)) \\
&= |a|(\cos\theta + i\sin\theta) \\
&= a
\end{aligned}$$

Again, two numbers $\cos\left(\frac{\theta+2k\pi}{n}\right)+i\sin\left(\frac{\theta+2k\pi}{n}\right)$ and $\cos\left(\frac{\theta+2m\pi}{n}\right)+i\sin\left(\frac{\theta+2m\pi}{n}\right)$ are equal if and only if $k-m$ is a multiple of n. Therefore, we conclude that the following n complex numbers:

$$\sqrt[n]{|a|}\left(\cos\left(\frac{\theta+2k\pi}{n}\right)+i\sin\left(\frac{\theta+2k\pi}{n}\right)\right)$$

are all the solutions to the root equation $z^n=a$. Similar to the case of roots of 1, we write the n-th root of a as:

Definition 1.9 (Roots of a Complex Number)**.** Given any $a\in\mathbb{C}\backslash\{0\}$ and $n\in\mathbb{N}$, the *n-th roots* of a is a set given by:

$$a^{\frac{1}{n}}=\left\{\sqrt[n]{|a|}\left(\cos\left(\frac{\mathrm{Arg}(a)+2k\pi}{n}\right)+i\sin\left(\frac{\mathrm{Arg}(a)+2k\pi}{n}\right)\right):k\in\{0,1,\ldots,n-1\}\right\}$$

Example 1.3. Find $i^{\frac{1}{3}}$ and $(1-\sqrt{3}i)^{\frac{1}{2}}$.

Solution

First express i into polar form $i=\cos\frac{\pi}{2}+i\sin\frac{\pi}{2}$. Hence by Definition 1.9, we have:

$$\begin{aligned}
i^{\frac{1}{3}}&=\left\{\cos\frac{\frac{\pi}{2}+2k\pi}{3}+i\sin\frac{\frac{\pi}{2}+2k\pi}{3}:k=0,1,2\right\}\\
&=\left\{\underbrace{\cos\frac{\pi}{6}+i\sin\frac{\pi}{6}}_{k=0},\ \underbrace{\cos\frac{5\pi}{6}+i\sin\frac{5\pi}{6}}_{k=1},\ \underbrace{\cos\frac{3\pi}{2}+i\sin\frac{3\pi}{2}}_{k=2}\right\}\\
&=\left\{\frac{\sqrt{3}+i}{2},\ \frac{\sqrt{3}-i}{2},\ -i\right\}
\end{aligned}$$

Similarly, to find $\left\{(1-\sqrt{3}i)^{\frac{1}{2}}\right\}$, we first express:

$$1-\sqrt{3}i=2\left(\cos\left(-\frac{\pi}{3}\right)+i\sin\left(-\frac{\pi}{3}\right)\right)$$

Hence, by Definition 1.9, we have:

$$\begin{aligned}
(1-\sqrt{3}i)^{\frac{1}{2}}&=\left\{\sqrt{2}\left(\cos\left(\frac{-\frac{\pi}{3}+2k\pi}{2}\right)+i\sin\left(\frac{-\frac{\pi}{3}+2k\pi}{2}\right)\right):k=0,1\right\}\\
&=\left\{\sqrt{2}\left(\frac{\sqrt{3}-i}{2}\right),\ \sqrt{2}\left(\frac{-\sqrt{3}+i}{2}\right)\right\}\\
&=\left\{\frac{\sqrt{3}-i}{\sqrt{2}},\ \frac{-\sqrt{3}+i}{\sqrt{2}}\right\}
\end{aligned}$$

Exercise 1.10. First, show that the roots of $z^4+1=0$ are:

$$\left\{\frac{1+i}{\sqrt{2}},\ \frac{1-i}{\sqrt{2}},\ \frac{-1+i}{\sqrt{2}},\ \frac{-1-i}{\sqrt{2}}\right\}.$$

Then, use this result to factorize z^4+1 into the product of two quadratic polynomials with real coefficients.

Exercise 1.11. By considering the roots of the equation $z^n - 1 = 0$ (where $n > 2$ is an integer), show that $z^n - 1$ can be factorized into a product of linear and quadratic polynomials with real coefficients:

$$z^n - 1 = \begin{cases} (z-1)(z+1)\prod_{r=1}^{k-1}\left(z^2 - 2z\cos\frac{2\pi r}{n} + 1\right) & \text{if } n = 2k \\ (z-1)\prod_{r=1}^{k-1}\left(z^2 - 2z\cos\frac{2\pi r}{n} + 1\right) & \text{if } n = 2k-1 \end{cases}$$

Next we discuss a useful observation about the n-th root of 1. Let

$$\omega = \cos\frac{2\pi}{n} + i\sin\frac{2\pi}{n}$$

where n is an integer with $n \geq 1$, then one can show the following identity holds:

$$1 + \omega + \omega^2 + \ldots + \omega^{n-1} = 0.$$

$$\begin{aligned}
&(1-\omega)(1+\omega+\omega^2+\ldots+\omega^{n-1}) \\
&= (1+\omega+\omega^2+\ldots+\omega^{n-1}) - \omega(1+\omega+\omega^2+\ldots+\omega^{n-1}) \\
&= (1+\omega+\omega^2+\ldots+\omega^{n-1}) - (\omega+\omega^2+\ldots+\omega^{n-1}+\omega^n) \\
&= 1-\omega^n \\
&= 1-\left(\cos\frac{2\pi}{n} + i\sin\frac{2\pi}{n}\right)^n \\
&= 1-(\cos(2\pi)+i\sin(2\pi)) = 1-1 = 0.
\end{aligned}$$

Since $\omega \neq 1$ as $n \geq 1$, we conclude that:

$$1 + \omega + \omega^2 + \ldots + \omega^{n-1} = 0.$$

Using this result, one can derive some trigonometric identities. Express ω in terms of its real and imaginary parts:

$$1 + \underbrace{\left(\cos\frac{2\pi}{n} + i\sin\frac{2\pi}{n}\right)}_{\omega} + \left(\cos\frac{2\pi}{n} + i\sin\frac{2\pi}{n}\right)^2 + \ldots + \left(\cos\frac{2\pi}{n} + i\sin\frac{2\pi}{n}\right)^{n-1} = 0$$

$$\begin{aligned}
1 + \left(\cos\frac{2\pi}{n} + i\sin\frac{2\pi}{n}\right) + \left(\cos\frac{4\pi}{n} + i\sin\frac{4\pi}{n}\right) + \ldots \\
+ \left(\cos\frac{2(n-1)\pi}{n} + i\sin\frac{2(n-1)\pi}{n}\right) = 0
\end{aligned}$$

By equating the real and imaginary parts, we obtain two trigonometric identities:

$$\begin{aligned}
\cos\frac{2\pi}{n} + \cos\frac{4\pi}{n} + \ldots + \cos\frac{2(n-1)\pi}{n} &= -1 \\
\sin\frac{2\pi}{n} + \sin\frac{4\pi}{n} + \ldots + \sin\frac{2(n-1)\pi}{n} &= 0
\end{aligned}$$

Exercise 1.12. Show that for any $z \neq 1$, we have

$$1 + z + z^2 + \ldots + z^n = \frac{1 - z^{n+1}}{1 - z},$$

and use it to show:

$$1 + \cos\theta + \cos 2\theta + \ldots + \cos n\theta = \frac{1}{2} + \frac{\sin\left(\frac{(2n+1)\theta}{2}\right)}{2\sin\frac{\theta}{2}}$$

for any $\theta \in (0, 2\pi)$.

Exercise 1.13. Let $n \geq 2$ be an integer.

(a) Solve the equation $(z+1)^n - 1 = 0$.

(b) Hence, show that $\sin\frac{\pi}{n} \cdot \sin\frac{2\pi}{n} \cdots \sin\frac{(n-1)\pi}{n} = \frac{n}{2^{n-1}}$.

(c) Consider a circle of radius 1, and let $P_1, P_2, \ldots, P_n$ be the vertices of a regular n-sided polygon inscribed in the circle. Denote the distance between any pair of points P and Q by $\overline{PQ}$. Using (b), show that:

$$\prod_{k=2}^{n} \overline{P_1 P_k} = n.$$

Exercise 1.14. Let $P_k(x_k, y_k)$, where $k = 1, 2, 3$, be three distinct points in $\mathbb{C}$ and let $z_k := x_k + y_k i$ be the complex number representing P_k. Denote $\omega = \cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}$. Show that $\triangle P_1P_2P_3$ is equilateral if and only if

$$z_1 + \omega z_2 + \omega^2 z_3 = 0.$$

Using this, show that it is impossible for $\triangle P_1P_2P_3$ being equilateral if $x_k, y_k \in \mathbb{Q}$ for all $k = 1, 2, 3$.

1.2. Sequences and Series

1.2.1. Sequences of Complex Numbers. In this section, we will extend the notion of sequences and series to complex numbers. As we shall see, many results and convergence tests which hold for real numbers will carry over to complex numbers. Let's begin with the definition of convergence of complex sequences:

Definition 1.10 (Limit of Sequences)**.** Let $\{z_n\}_{n=1}^{\infty}$ be a sequence of complex numbers. We say z_n *converges to* w *as* $n \to \infty$ if for any $\varepsilon > 0$, there exists an integer $N > 0$ such that whenever $n \geq N$, we have $|z_n - w| < \varepsilon$.

Remark 1.11. We may abbreviate "z_n converges to w as $n \to \infty$" by simply saying:

$$\lim_{n\to\infty} z_n = w.$$

Remark 1.12. It is easy to see that $\lim_{n\to\infty} z_n = w$ is equivalent to $\lim_{n\to\infty} |z_n - w| = 0$.

Remark 1.13. The definition of convergence of complex sequences is almost the same as the that of real sequences. The only difference is now $|\cdot|$ represents the modulus while for real sequence it represents the absolute value. Therefore, many computational rules about limits carry over to complex sequences. For instance, if $\lim_{n\to\infty} z_n = L$ and $\lim_{n\to\infty} w_n = M$, then we have

$$\begin{aligned} \lim_{n\to\infty} (z_n \pm w_n) &= L \pm M \\ \lim_{n\to\infty} (z_n w_n) &= LM \\ \lim_{n\to\infty} \frac{z_n}{w_n} &= \frac{L}{M} && \text{(whenever } M \neq 0\text{)} \end{aligned}$$

Example 1.4. Consider the sequence $z_n = z^n$ where $z \in \mathbb{C}$ is a fixed complex number. Show from the definition of limits that:

- if $|z| < 1$, then z_n converges to 0 as $n \to \infty$;
- if $z = 1$, then z_n converges to 1 as $n \to \infty$;

Solution

First consider the case $|z| < 1$: if $z = 0$, then $z_n = 0$ for any n and the desired result clearly holds. From now on we assume $z \neq 0$. For any $\varepsilon > 0$, we pick a positive integer $N > \frac{\log \varepsilon}{\log|z|}$. Whenever, $n \geq N$, we have:

$$|z_n - 0| = |z^n| = |z|^n \leq |z|^N.$$

Here we have used the fact that $|z| < 1$ and $n \geq N$. By our choice of N, we have:

$$\begin{aligned} |z|^N &< |z|^{\frac{\log \varepsilon}{\log|z|}} \\ &= |z|^{\log_{|z|} \varepsilon} = \varepsilon. \end{aligned}$$

This shows $\lim_{n\to 0} z_n = 0$ in case of $|z| < 1$. The case of $z = 1$ is trivial.

When $|z| \geq 1$ and $z \neq 1$, the sequence $z_n = z^n$ can be shown to diverge using the squeezing principle (see Exercise 1.16). It can also be proved using the following useful fact:

Proposition 1.14. *A sequence $\{z_n\} \in \mathbb{C}$ converges to w as a complex sequence if and only if $\{\mathrm{Re}(z_n)\}$ converges to $\mathrm{Re}(w)$ and $\{\mathrm{Im}(z_n)\}$ converges to $\mathrm{Im}(w)$ as real sequences.*

Proof. ($\Longrightarrow$)-part follows from the inequalities:

$$|\mathrm{Re}(z_n) - \mathrm{Re}(w)| \leq |z_n - w| \quad \text{and} \quad |\mathrm{Im}(z_n) - \mathrm{Im}(w)| \leq |z_n - w|$$

and the squeezing principle.

($\Longleftarrow$)-part follows from the fact that:

$$|z_n - w| = \sqrt{|\mathrm{Re}(z_n) - \mathrm{Re}(w)|^2 + |\mathrm{Im}(z_n) - \mathrm{Im}(w)|^2}$$

□

Now given a complex number z expressed in polar form as $z = r(\cos\theta + i\sin\theta)$, and suppose $|z| \geq 1$ (i.e. $r \geq 1$) and $z \neq 1$. Consider again the sequence $z_n = z^n$. By De Moivre's Theorem, we have:

$$z_n = r^n(\cos n\theta + i\sin n\theta).$$

It is well known in real analysis that when $\theta \neq 2k\pi$ (where $k \in \mathbb{Z}$), at least one of the real sequences $\{\cos n\theta\}$ and $\{\sin n\theta\}$ diverges as $n \to \infty$. Hence, when $r \geq 1$ and $\theta \neq 2k\pi$ ($k \in \mathbb{Z}$), at least one of the real sequences $\{r^n \cos n\theta\}$ and $\{r^n \sin n\theta\}$ diverges. This shows z_n diverges.

Exercise 1.15. Show that if $\lim_{n\to\infty} z_n = L$, then $\lim_{n\to\infty} \overline{z_n} = \overline{L}$ and $\lim_{n\to\infty} |z_n| = |L|$.

Exercise 1.16. Show (without using Proposition 1.14) that if $|z| \geq 1$ and $z \neq 1$, then the sequence $\{z^n\}$ must diverge. [*Hint: First prove the following inequality:*

$$|z - 1| \leq \left|z^{n+1} - w\right| + |z^n - w|$$

for any $z \in \mathbb{C}$ such that $|z| \geq 1$, and any $w \in \mathbb{C}$.]

In Real Analysis, there is a notion of *Cauchy sequences* which describe sequences that are closer and closer to each other. It is *a priori* different from *convergent sequences*, which are sequences that are closer and closer to a certain limit. However, it is well-known that for sequences in $\mathbb{R}$, the Cauchy condition will guarantee convergence. This important fact is known as *completeness of real numbers*.

In Complex Analysis, we have a similar notion of *Cauchy sequences* and *completeness*, to be discussed below.

Definition 1.15 (Cauchy Sequence)**.** A sequence $\{z_n\}_{n=1}^{\infty}$ of complex numbers is said to be a *Cauchy sequence* if and only if for any $\varepsilon > 0$, there exists an integer $N \in \mathbb{N}$ such that whenever $m, n \geq N$, we have $|z_n - z_m| < \varepsilon$.

Theorem 1.16 (Completeness of $\mathbb{C}$)**.** *Every Cauchy sequence of complex numbers converges to a certain complex number. In other words, $\mathbb{C}$ is complete.*

Proof. Let $\{z_n\}$ be a Cauchy sequence of complex numbers. We need to show it converges. Write $z_n = x_n + iy_n$, where $x_n, y_n \in \mathbb{R}$. Since we have:

$$|x_n - x_m| \leq |z_n - z_m|$$
$$|y_n - y_m| \leq |z_n - z_m|$$

and given that $\{z_n\}$ is a Cauchy sequence, the real sequences $\{x_n\}$ and $\{y_n\}$ are also Cauchy sequences. By Completeness of $\mathbb{R}$, both $\{x_n\}$ and $\{y_n\}$ converge to some real numbers x_∞ and y_∞ respectively. By Proposition 1.14, the complex sequence $\{z_n\}$ converges to $x_\infty + iy_\infty$. □

Exercise 1.17. Suppose $\{z_n\}_{n=0}^{\infty}$ is a complex sequence. Suppose there exists a real constant $\alpha \in [0, 1)$ such that:

$$|z_{n+1} - z_n| \leq \alpha |z_n - z_{n-1}| \quad \text{for any } n \in \mathbb{N}.$$

Show that the complex sequence $\{z_n\}_{n=0}^{\infty}$ converges.

1.2.2. Series of Complex Numbers. An (infinite) series $\sum_{n=1}^{\infty} z_n$ of complex numbers $z_n \in \mathbb{C}$ is the limit (if exists) of the N-th partial sums $\sum_{n=1}^{N} z_n$ as $N \to \infty$. In Real Analysis, we learned that many series convergence tests rely on the fact that $\mathbb{R}$ is complete. Now that we know $\mathbb{C}$ is also complete (Theorem 1.16), we can generalize many (although not all) series convergence tests for $\mathbb{C}$.

Definition 1.17 (Absolute and Conditional Convergences)**.** A series of complex numbers $\sum_{n=1}^{\infty} z_n$ is said to converge *absolutely* if the series $\sum_{n=1}^{\infty} |z_n|$ converges. A series $\sum_{n=1}^{\infty} z_n$ is said to converge *conditionally* if it converges but does not converge absolutely.

Proposition 1.18 (Absolute Convergence Test)**.** *If the series $\sum_{n=1}^{\infty} |z_n|$ converges, then the complex series $\sum_{n=1}^{\infty} z_n$ also converges.*

Proof. Given that $\sum_{n=1}^{\infty} |z_n|$ converges, its N-th partial sum $\sum_{n=1}^{N} |z_n|$ is a Cauchy sequence. Now consider the sequence of N-th partial sums $\sum_{n=1}^{N} z_n$. We want to show the later is also a Cauchy sequence.

For any $\varepsilon > 0$, there exists an integer $K > 0$ such that whenever $M > N \geq K$, we have

$$\sum_{n=1}^{M} |z_n| - \sum_{n=1}^{N} |z_n| < \varepsilon.$$

It implies:

$$\left| \sum_{n=1}^{M} z_n - \sum_{n=1}^{N} z_n \right| = \left| \sum_{n=N+1}^{M} z_n \right| \leq \sum_{n=N+1}^{M} |z_n| = \sum_{n=1}^{M} |z_n| - \sum_{n=1}^{N} |z_n| < \varepsilon.$$

Therefore, $\sum_{n=1}^{N} z_n$ is also a Cauchy sequence. By completeness of $\mathbb{C}$ (Theorem 1.16), the N-th partial sum $\sum_{n=1}^{N} z_n$ (and hence the infinite series $\sum_{n=1}^{\infty} z_n$) converges. □

Example 1.5. Does the series $\sum_{n=1}^{\infty} \frac{i^n}{n}$ converge absolutely, conditionally, or does not converge? How about the series $\sum_{n=1}^{\infty} \frac{i^n}{n^2}$?

Solution

The series $\sum_{n=1}^{\infty} \left|\frac{i^n}{n}\right| = \sum_{n=1}^{\infty} \frac{1}{n}$ diverges by p-test. The N-th partial sum can be decomposed into:

$$\sum_{n=1}^{N} \frac{i^n}{n} = \begin{cases} \left(-\frac{1}{2}+\frac{1}{4}-\ldots+\frac{(-1)^k}{2k}\right)+\left(1-\frac{1}{3}+\frac{1}{5}+\ldots+\frac{(-1)^{k-1}}{2k-1}\right)i & \text{if } N=2k \\ \left(-\frac{1}{2}+\frac{1}{4}-\ldots+\frac{(-1)^k}{2k}\right)+\left(1-\frac{1}{3}+\frac{1}{5}+\ldots+\frac{(-1)^{k+1}}{2k+1}\right)i & \text{if } N=2k+1 \end{cases}$$

In either case, the real and imaginary parts converge by alternating series test. By Proposition 1.14, the series $\sum_{n=1}^{\infty} \frac{i^n}{n}$ converges, and so it converges conditionally.

Now consider $\sum_{n=1}^{\infty} \frac{i^n}{n^2}$. The series $\sum_{n=1}^{\infty} \left|\frac{i^n}{n^2}\right| = \sum_{n=1}^{\infty} \frac{1}{n^2}$ converges by p-test. Therefore, the series $\sum_{n=1}^{\infty} \frac{i^n}{n^2}$ converges absolutely.

One good property of an absolute convergent series is that we can *rearrange* the terms as we wish without changing the value of the series. Precisely, given an absolute convergent series $\sum_{n=1}^{\infty} z_n =: L$ and a bijection $\sigma : \mathbb{N} \to \mathbb{N}$, then the rearranged series $\sum_{n=1}^{\infty} z_{\sigma(n)}$ also converges absolutely to the limit L. The proof is the same as in the real case (hence omitted here).

Recall from Real Analysis that the ratio test and root test follow from the absolute convergence test and completeness of $\mathbb{R}$. Now we learned that both hold on $\mathbb{C}$, hence the ratio test and root test can be extended to complex series:

Proposition 1.19 (Ratio Test). *Consider the complex series* $\sum_{n=1}^{\infty} z_n$*:*

- *If* $\lim_{n\to\infty} \left|\frac{z_{n+1}}{z_n}\right| < 1$, *then* $\sum_{n=1}^{\infty} z_n$ *converges absolutely.*
- *If* $\lim_{n\to\infty} \left|\frac{z_{n+1}}{z_n}\right| > 1$, *then* $\sum_{n=1}^{\infty} z_n$ *diverges.*
- *If* $\lim_{n\to\infty} \left|\frac{z_{n+1}}{z_n}\right| = 1$, *then no conclusion can be drawn.*

Proposition 1.20 (Root Test)**.** *Consider the complex series* $\sum_{n=1}^{\infty} z_n$*:*

- *If* $\lim_{n\to\infty} \sqrt[n]{|z_n|} < 1$*, then* $\sum_{n=1}^{\infty} z_n$ *converges absolutely.*
- *If* $\lim_{n\to\infty} \sqrt[n]{|z_n|} > 1$*, then* $\sum_{n=1}^{\infty} z_n$ *diverges.*
- *If* $\lim_{n\to\infty} \sqrt[n]{|z_n|} = 1$*, then no conclusion can be drawn.*

Remark 1.21. The proofs of the ratio and root tests are the same as in the real case. We omit their proofs but we encourage readers to write down their proofs as an exercise.

Example 1.6. Show that for any $z \in \mathbb{C}$, the complex series $\sum_{n=0}^{\infty} \frac{z^n}{n!}$ converges absolutely.

Solution

We use the ratio test. Consider:

$$\begin{aligned}\lim_{n\to\infty} \left| \frac{z^{n+1}/(n+1)!}{z^n/n!} \right| &= \lim_{n\to\infty} \left| \frac{z^{n+1}}{z^n} \frac{n!}{(n+1)!} \right| \\ &= \lim_{n\to\infty} \left| \frac{z}{n+1} \right| = \lim_{n\to\infty} \frac{|z|}{n+1} \\ &= 0 < 1 \quad \text{for any } z \in \mathbb{C}.\end{aligned}$$

Hence, the series $\sum_{n=0}^{\infty} \frac{z^n}{n!}$ converges absolutely by ratio test (Proposition 1.19).

Alternatively, we can also use the root test (Proposition 1.20) by showing that:

$$\lim_{n\to\infty} \sqrt[n]{\left| \frac{z^n}{n!} \right|} = \lim_{n\to\infty} \frac{|z|}{\sqrt[n]{n!}} = 0 < 1$$

for any $z \in \mathbb{C}$. Here we have used the fact that $\lim_{n\to\infty} \sqrt[n]{n!} = \infty$.

Example 1.7. Determine all complex numbers z such that the series $\sum_{n=0}^{\infty} nz^n$ converges.

Solution

Consider the limit $\lim_{n\to\infty} \left| \frac{(n+1)z^{n+1}}{nz^n} \right| = \lim_{n\to\infty} \frac{(n+1)}{n} |z| = |z|$. Therefore, by ratio test (Proposition 1.19), the series converges absolutely when $|z| < 1$; and diverges when $|z| > 1$.

When $|z| = 1$, the ratio test fails to conclude anything. In this case, we let $z = \cos\theta + i\sin\theta$ where $\theta \in \mathbb{R}$. Then, the series is given by $\sum_{n=0}^{\infty} (n\cos n\theta + in\sin n\theta)$, and the real and imaginary parts are:

$$\operatorname{Re}\left(\sum_{n=0}^{\infty} nz^n\right) = \sum_{n=0}^{\infty} n\cos(n\theta) \quad \text{and} \quad \operatorname{Im}\left(\sum_{n=0}^{\infty} nz^n\right) = \sum_{n=0}^{\infty} n\sin(n\theta).$$

By Proposition 1.14, if the complex series converges, then both their real and imaginary parts converge, and in particular we have:

$$\lim_{n\to\infty} n\cos(n\theta) = 0 \quad \text{and} \quad \lim_{n\to\infty} n\sin(n\theta) = 0.$$

By Squeeze Theorem, it will imply:

$$\lim_{n\to\infty} \cos(n\theta) = \lim_{n\to\infty} \sin(n\theta) = 0.$$

However, it would contradict the fact that $\cos^2(n\theta) + \sin^2(n\theta) = 1$; and so the series $\sum_{n=0}^{\infty} nz^n$ does not converge when $|z| = 1$.

Conclusion: the series $\sum_{n=0}^{\infty} nz^n$ converges if and only if $|z| < 1$.

Exercise 1.18. Determine whether each of the following complex series converges absolutely, conditionally, or diverge:

(a) $\sum_{n=0}^{\infty} \frac{(1-3i)^n}{(4+i)^{2n}}$

(b) $\sum_{n=1}^{\infty} \frac{n^2}{n+n^3 i}$

(c) $\sum_{n=1}^{\infty} (\cos n - i\sin n)$

Exercise 1.19. In each of the following complex series: (i) determine all complex numbers z such that the series converges, (ii) sketch the range of these z's on the complex plane $\mathbb{C}$.

(a) $\sum_{n=1}^{\infty} z^n$

(b) $\sum_{n=1}^{\infty} \left(\frac{z}{z+1}\right)^n$

(c) $\sum_{n=1}^{\infty} \frac{(-1)^n z^{5n}}{n!}$

(d) $\sum_{n=1}^{\infty} \frac{z^{n!}}{n^2}$

Exercise 1.20. Suppose $z \in \mathbb{C}$.

(a) Assume $|z| \neq 1$ and $z \neq 0$, show that for any $n \in \mathbb{N}$, we have:
$$\left|\frac{z^n}{1+z^{2n}}\right| \leq \frac{1}{\left||z|^n - |z|^{-n}\right|}$$

(b) Using (a), or otherwise, find all $z \in \mathbb{C}$ such that the sequence $\left\{\frac{z^n}{1+z^{2n}}\right\}_{n=1}^{\infty}$ converges.

(c) Find all $z \in \mathbb{C}$ such that the series $\sum_{n=1}^{\infty} \frac{z^n}{1+z^{2n}}$ converges.

1.2.3. Euler's Identity. The series $\sum_{n=0}^{\infty} \frac{z^n}{n!}$ considered in Example 1.6 is an important one – it *defines* the complex exponential function. When $z = x$ is a real number, the value of the series is given by e^x. Given that the series converges for any $z \in \mathbb{C}$, we define e^z to be the limit of this series:

Definition 1.22 (Complex Exponential)**.** Let $z \in \mathbb{C}$, the exponential e^z, or equivalently $\exp(z)$, of z is defined by:
$$e^z := \sum_{n=0}^{\infty} \frac{z^n}{n!}.$$

Remark 1.23. Please do NOT ask *why* $e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!}$, as it is by the *definition*. A more appropriate question is *what* motivates such a definition. One motivation is that by such a definition, many nice properties about e^x in the real case can be extended to e^z in the complex case. These properties may include $e^{z+w} = e^z e^w$, $e^z \neq 0$, etc. We will look into them soon.

Here is the famous Euler's identity that relates complex exponentials with the polar form of a complex number:

Theorem 1.24 (Euler's Identity)**.** *For any $\theta \in \mathbb{R}$, we have:*

(1.3)
$$e^{i\theta} = \cos\theta + i\sin\theta.$$

Proof. The key idea is to split the defining series into real and imaginary parts.

$$\begin{aligned}
e^{i\theta} &= \sum_{n=0}^{\infty} \frac{(i\theta)^n}{n!} = \lim_{N\to\infty} \sum_{n=0}^{2N} \frac{i^n\theta^n}{n!} \\
&= \lim_{N\to\infty} \left(\sum_{k=0}^{N} \frac{i^{2k}\theta^{2k}}{(2k)!} + \sum_{k=0}^{N-1} \frac{i^{2k+1}\theta^{2k+1}}{(2k+1)!} \right) && \text{[by rearrangement]} \\
&= \underbrace{\lim_{N\to\infty} \sum_{k=0}^{N} \frac{(-1)^k\theta^{2k}}{(2k)!}}_{=\cos\theta} + i \underbrace{\left(\lim_{N\to\infty} \sum_{k=0}^{N-1} \frac{(-1)^k\theta^{2k+1}}{(2k+1)!} \right)}_{=\sin\theta} && \text{[using } i^{2k} = (i^2)^k = (-1)^k\text{]} \\
&= \cos\theta + i\sin\theta
\end{aligned}$$

□

Remark 1.25. From (1.3), it is evident that we have:

$$e^{i\pi} + 1 = 0$$

which is a single identity involving 5 most important constants in mathematics, namely 1, 0, e, π and i.

Remark 1.26. From the Euler's identity, we can now write down the polar form of a complex number in a simpler way: if $z = r(\cos\theta + i\sin\theta)$, then we can write:

$$z = re^{i\theta}.$$

In particular, any $z \in \mathbb{C}$ such that $|z| = 1$ can be expressed as $z = e^{i\theta}$ for some $\theta \in \mathbb{R}$.

We are going to show that the complex exponential has the property that $e^z e^w = e^{z+w}$ just like the real case. *Informally*, we express both e^z and e^w into two infinite series. After multiplying the two series, we express the double sum diagonally:

$$\begin{aligned}
e^z e^w &= \left(\sum_{n=0}^{\infty} \frac{z^n}{n!}\right)\left(\sum_{m=0}^{\infty} \frac{w^m}{m!}\right) = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \frac{z^n w^m}{n!m!} \\
&= \sum_{k=0}^{\infty}\sum_{m+n=k} \frac{z^n w^m}{n!m!} = \sum_{k=0}^{\infty}\sum_{n=0}^{k} \frac{z^n w^{k-n}}{n!(k-n)!} && [\text{since } m = k-n] \\
&= \sum_{k=0}^{\infty}\sum_{n=0}^{k} \frac{C_n^k z^n w^{k-n}}{k!} && [\text{since } C_n^k = \frac{k!}{(n-k)!n!}] \\
&= \sum_{k=0}^{\infty} \frac{(z+w)^k}{k!} = e^{z+w} && [\text{Binomial Theorem}]
\end{aligned}$$

Although this "proof" above seems convincing and neat, there is a little step we need to justify, namely why we can rearrange the *infinite* double sum $\sum_n \sum_m$ in a diagonal way: $\sum_k \sum_{m+n=k}$? We have seen in Real Analysis that even switching $\sum_n$ and $\sum_m$ may sometimes result in a different sum. Below we give a rigorous (and more refined) proof of this fact:

Proposition 1.27. *For any $z, w \in \mathbb{C}$, we have $e^z e^w = e^{z+w}$.*

Proof. Consider the N-th partial sums $\displaystyle\sum_{n=0}^{N} \frac{z^n}{n!}$ and $\displaystyle\sum_{m=0}^{N} \frac{w^m}{m!}$, then:

$$\begin{aligned}
\left(\sum_{n=0}^{N} \frac{z^n}{n!}\right)\left(\sum_{m=0}^{N} \frac{w^m}{m!}\right) &= \underbrace{\sum_{n=0}^{N}\sum_{m=0}^{N} \frac{z^n w^m}{n!m!}}_{\text{Region I in Fig. 1.2}} \\
&= \underbrace{\sum_{k=0}^{2N}\sum_{m+n=k} \frac{z^n w^m}{n!m!}}_{\text{Region I+II+III in Fig. 1.2}} - \underbrace{\sum_{m=0}^{N}\sum_{n=N+1}^{2N-m+1} \frac{z^n w^m}{n!m!}}_{\text{Region II in Fig. 1.2}} - \underbrace{\sum_{n=0}^{N}\sum_{m=N+1}^{2N-n+1} \frac{z^n w^m}{n!m!}}_{\text{Region III in Fig. 1.2}}
\end{aligned}$$

Here we break down the finite double sum $\sum_n^N \sum_m^N$ into three triangular sums. See Figure 1.2 for illustration. For the sum corresponding to the large triangle (Region I+II+III in Figure 1.2), we can rewrite it as:

$$\sum_{k=0}^{2N}\sum_{m+n=k} \frac{z^n w^m}{n!m!} = \sum_{k=0}^{2N}\sum_{n=0}^{k} \frac{z^n w^{k-n}}{n!(k-n)!} = \sum_{k=0}^{2N}\sum_{n=0}^{k} \frac{C_n^k z^n w^{k-n}}{k!} = \sum_{k=0}^{2N} \frac{(z+w)^k}{k!} \to e^{z+w}$$

as $N \to \infty$.

For Region II in Figure 1.2, we can show that it converges to 0 as $N \to \infty$:

$$\begin{aligned}\left|\sum_{m=0}^{N}\sum_{n=N+1}^{2N-m+1}\frac{z^n w^m}{n!m!}\right| &\leq \sum_{m=0}^{N}\sum_{n=N+1}^{2N-m+1}\left|\frac{z^n w^m}{n!m!}\right| \\ &\leq \sum_{m=0}^{N}\sum_{n=N+1}^{\infty}\frac{|z|^n}{n!}\frac{|w|^m}{m!} \\ &= \left(\sum_{m=0}^{N}\frac{|w|^m}{m!}\right)\left(\sum_{n=N+1}^{\infty}\frac{|z|^n}{n!}\right) \\ &\leq e^{|w|}\left(\sum_{n=N+1}^{\infty}\frac{|z|^n}{n!}\right) \to 0\end{aligned}$$

as $N \to \infty$ since the infinite sum $\sum_{n=0}^{\infty}\frac{|z|^n}{n!}$ converges (to $e^{|z|}$). The sum corresponding to Region III in Figure 1.2 can be shown to converge to 0 by switching m and n, and z and w in the above argument.

Overall, we have shown:

$$\begin{aligned}&\lim_{N\to\infty}\left(\sum_{n=0}^{N}\frac{z^n}{n!}\right)\left(\sum_{m=0}^{N}\frac{w^m}{m!}\right) \\ &= \lim_{N\to\infty}\sum_{k=0}^{2N}\sum_{m+n=k}\frac{z^n w^m}{n!m!} - \lim_{N\to\infty}\sum_{m=0}^{N}\sum_{n=N+1}^{2N-m+1}\frac{z^n w^m}{n!m!} - \lim_{N\to\infty}\sum_{n=0}^{N}\sum_{m=N+1}^{2N-n+1}\frac{z^n w^m}{n!m!} \\ &= e^{z+w} - 0 - 0,\end{aligned}$$

which implies $e^z e^w = e^{z+w}$ as desired. □

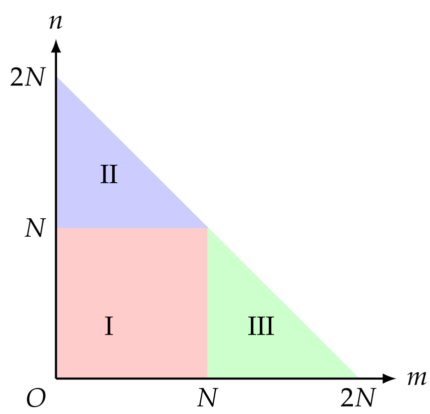


Figure 1.2

Exercise 1.21. Given two series $\sum_{n=0}^{\infty} z_n$ and $\sum_{n=0}^{\infty} w_n$ which converge absolutely to A and B respectively, show that the series below converges *absolutely* to AB:

$$\sum_{k=0}^{\infty}\left(\sum_{n=0}^{k} z_n w_{k-n}\right)$$

Exercise 1.22. Suppose $\{a_n\}_{n=1}^{\infty}$ is a (monotonically) decreasing sequence of *real* numbers such that $\lim_{n\to\infty} a_n = 0$, and $\{z_n\}_{n=1}^{\infty}$ be a sequence of complex numbers with the property that there is a constant $C > 0$ such that $\left|\sum_{n=1}^{N} z_n\right| \leq C$ for any N. Show that the series $\sum_{n=1}^{\infty} a_n z_n$ converges. [*Hint: First prove the following summation-by-parts formula*

$$\sum_{n=1}^{N} a_n z_n = \sum_{n=1}^{N} a_{N+1} z_n + \sum_{n=1}^{N} \sum_{k=1}^{n} (a_n - a_{n+1}) z_k,$$

and make good use of the given conditions.]

Furthermore, use the above result to prove the alternating series test in Real Analysis.

Exercise 1.23. Using the result from Exercise 1.22, show that the series $\sum_{n=1}^{\infty} \frac{z^n}{n}$ converges for any z such that $|z| = 1$ and $z \neq 1$.

Using the multiplicative property $e^z e^w = e^{z+w}$, one can show the following properties about the complex exponential function. We leave the proofs for readers.

Remark 1.28. For any $z = x + yi \in \mathbb{C}$ where $x, y \in \mathbb{R}$, we have:

- $(e^z)^n = e^{nz}$ for any integer n.
- $e^z = e^x e^{iy} = e^x(\cos y + i \sin y)$, and hence $|e^z| = e^x$.
- $e^z \neq 0$.

The complex exponential a^z with other real base $a > 0$ is defined via the natural exponential e^z. Recall that $a = e^{\ln a}$, and we define:

$$a^z := e^{(\ln a)z}.$$

Using this definition, some properties of e^z extend to complex exponentials a^z with an arbitrary real base $a > 0$. Proofs are again left for readers.

Remark 1.29. For any real $a, b > 0$ and $z, w \in \mathbb{C}$ we have:

- $(a^z)^n = a^{nz}$ for any integer n.
- $a^z a^w = a^{z+w}$
- $|a^z| = a^{\mathrm{Re}(a)}$
- $a^z \neq 0$
- $(ab)^z = a^z b^z$

Remark 1.30. For any positive integer n, the rational number $\frac{1}{n}$ is no doubt also a complex number. Therefore, now $e^{\frac{1}{n}}$ could mean two different things, namely the value of the series $\sum_{k=0}^{\infty} \frac{\left(\frac{1}{n}\right)^k}{k!}$, or the n-th roots of e. It is a confusing ambiguity but fortunately we seldom deal with both of them in the same context. One way to avoid such a confusion is to represent the n-th roots of e by $e^{\frac{1}{n}}$, and use $\exp(\frac{1}{n})$ to represent the value of the aforesaid series.

1.2.4. Riemann ζ Function: the first encounter. The Riemann zeta function, denoted by $\zeta(z)$, is of central importance in Complex Analysis and Number Theory. It is an infinite series defined as:

$$\zeta(z) := \sum_{n=1}^{\infty} \frac{1}{n^z}$$

for $\operatorname{Re}(z) > 1$. This complex series motivates the discussions of the famous *Riemann Hypothesis*, which is a conjecture purposed by Riemann in 1859 and remains unsolved as of today (January 20, 2017). The statement of the Riemann Hypothesis will be explained after we learn about analytic continuation of holomorphic functions. The Riemann zeta function has deep connections with Number Theory, in particular on the study of distribution of prime numbers. It is used to show the renowned *Prime Number Theorem*, which asserts that:

$$\lim_{x \to \infty} \frac{\pi(x)}{x / \ln x} = 1$$

where $\pi(x)$ is the number of positive prime numbers less than or equal to x.

The deep connection between $\zeta(z)$ and prime numbers is beyond the scope of this course. Meanwhile, we would like to point out that this series converges absolutely when $\operatorname{Re}(z) > 1$ by the (real) p-test. The main reason is as follows. Write $z = x + yi$ where $x, y \in \mathbb{R}$, then we have:

$$\left| \frac{1}{n^z} \right| = \left| \frac{1}{e^{z \log n}} \right| = \frac{1}{\left| e^{x \log n} e^{iy \log n} \right|} = \frac{1}{n^x} = \frac{1}{n^{\operatorname{Re}(z)}}.$$

By (real) p-test, the series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^{\operatorname{Re}(z)}}$ converges if and only if $\operatorname{Re}(z) > 1$. Therefore, by the (complex) absolute convergence test, the series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^z}$ converges absolutely when $\operatorname{Re}(z) > 1$.

1.3. Point-Set Topology of $\mathbb{C}$

In this section, we will introduce several terminologies and topological concepts about subsets of $\mathbb{C}$. These topological concepts will come up from time to time in the course.

To begin, let's define some standard notations we will use in the rest of the course. Let $z_0 \in \mathbb{C}$ and $r > 0$. From now on, we will denote:

$$B_r(z_0) = \{z \in \mathbb{C} : |z - z_0| < r\}$$
$$\overline{B_r(z_0)} = \{z \in \mathbb{C} : |z - z_0| \leq r\}$$
$$\partial B_r(z_0) = \{z \in \mathbb{C} : |z - z_0| = r\}$$

which are respectively the open ball, closed ball and circle with radius r centered at z_0. In the literature of Complex Analysis, it is often that the term *disc* is used instead of *ball*.

1.3.1. Open and Closed Subsets.

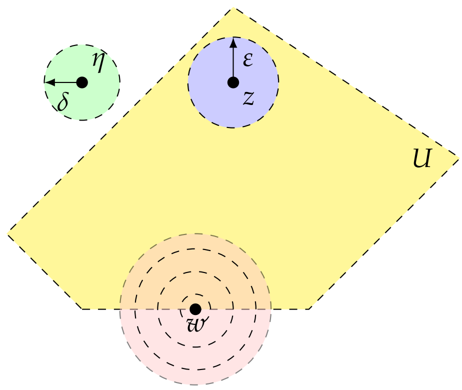
Intuitively, an open subset Ω in $\mathbb{C}$ is one that does not have a boundary. However, this "definition" is not rigorous enough since the term "boundary" has not been defined so far. We are going to give a rigorous definition of open and closed subsets here. We first define:

Definition 1.31 (Interior, Boundary and Exterior Points)**.** Consider a set $U \subset \mathbb{C}$. We say that $z \in \mathbb{C}$ is an *interior point* of U if there exists $\varepsilon > 0$ such that $B_\varepsilon(z) \subset U$. We say that $w \in \mathbb{C}$ is a *boundary point* of U if for any $\varepsilon > 0$, both $B_\varepsilon(w) \cap U$ and $B_\varepsilon(w) \cap (\mathbb{C} \backslash U)$ are non-empty. We say $\eta \in \mathbb{C}$ is an *exterior point* of U if there exists $\delta > 0$ such that $B_\delta(\eta) \subset \mathbb{C} \backslash U$.

In the figure below, the yellow set is the subset $U \subset \mathbb{C}$. The point $z \in U$ is an interior ball because by drawing a ball with a small enough radius (i.e. the blue ball), the ball is completely inside U. In *layman* terms, an interior point of U is a point z whose "nearby" points are contained in U.

On the other hand, the point w in the figure below is a boundary point. No matter how small the ball you draw around w, that ball must contain some points in U, and some points not in U. In *layman* terms, a boundary point of U is a point w at which if you look around it, you can see "nearby" some points in U and some point not in U.

The point η in the figure is an exterior point of U. In *layman* terms, it is a point whose "nearby" are outside U.



Remark 1.32. Since $z \in B_\varepsilon(z)$ for any $z \in \mathbb{C}$ and $\varepsilon > 0$, if z is an interior point of U, it is automatically that $z \in U$. In other words, an interior point of a set must belong to that set. However, a boundary point of a set can be contained or not contained in the set. Furthermore, according to the definitions, interior points, boundary points and exterior points are mutually exclusive.

Example 1.8. Find all interior, boundary and exterior points of the set:

$$U = \{z \in \mathbb{C} : \text{Re}(z) > 1\}.$$

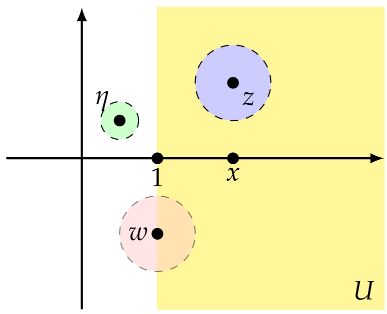
Solution

We claim that the set of interior points is U itself. For any $z \in U$, we have $\text{Re}(z) > 1$. Write $z = x + yi$, then we have $x > 1$. We need to find a small $\varepsilon > 0$ such that $B_\varepsilon(x + yi) \subset U$. According to the figure below, an appropriate choice of ε should be $\varepsilon = \frac{x-1}{2}$. We next verify that it is indeed $B_\varepsilon(z) \subset U$ for this choice of ε.

For any $\alpha \in B_\varepsilon(z)$, we have $|\alpha - z| < \varepsilon = \dfrac{x-1}{2}$. Then, by $\text{Re}(z - \alpha) \leq |z - \alpha|$, we know that:

$$\text{Re}(z - \alpha) < \frac{x-1}{2} \quad \Longrightarrow \quad x - \text{Re}(\alpha) < \frac{x-1}{2}.$$

By rearrangement, we get $\text{Re}(\alpha) > x - \dfrac{x-1}{2} = \dfrac{x+1}{2} > \dfrac{1+1}{2} = 1$, which is equivalently to saying that $\alpha \in U$. This shows $B_\varepsilon(z) \subset U$, and hence z is an interior point.



Next we show that every point w with $\text{Re}(w) = 1$ is a boundary point of U. Given any $\varepsilon > 0$, we consider the ball $B_\varepsilon(w)$. The point $w - \frac{\varepsilon}{2}$ lies in the ball $B_\varepsilon(w)$ and has real part $1 - \frac{\varepsilon}{2}$ and hence is not in U; while the point $w + \frac{\varepsilon}{2}$ is also in the ball $B_\varepsilon(w)$ but has real part $1 + \frac{\varepsilon}{2}$ and so is inside U. Therefore, both $B_\varepsilon(w) \cap U$ and $B_\varepsilon(w) \cap (\mathbb{C}\backslash U)$ are non-empty, it concludes that w is a boundary point of U.

Finally, we claim that any point $\eta \in \mathbb{C}$ with $\text{Re}(\eta) < 1$ is an exterior point of U. To prove this claim, we choose a $\delta = \frac{1-\text{Re}(\eta)}{2}$ and show that $B_\delta(\eta) \subset \mathbb{C}\backslash U$: Given any $\beta \in B_\delta(\eta)$, we have:

$$\text{Re}(\beta - \eta) \leq |\beta - \eta| < \delta = \frac{1 - \text{Re}(\eta)}{2} \quad \Longrightarrow \quad \text{Re}(\beta) < \frac{1 + \text{Re}(\eta)}{2} < 1.$$

Therefore, $\beta \notin U$, and it shows $B_\delta \subset \mathbb{C}\backslash U$. It completes the claim that η is an exterior point of U.

Exercise 1.24. Find all the interior, boundary and exterior points of each set below:

(a) $U_1 = \{z \in \mathbb{C} : \text{Re}(z) \geq 0 \text{ and } \text{Im}(z) < 0\}$.

(b) $U_2 = B_r(z_0)$ where $z_0 \in \mathbb{C}$ is a fixed number and $r > 0$

(c) $U_3 = \overline{B_r(z_0)}$ where $z_0 \in \mathbb{C}$ is a fixed number and $r > 0$.

(d) $U_4 = \partial B_r(z_0)$ where $z_0 \in \mathbb{C}$ is a fixed number and $r > 0$.

(e) $U_5 = \mathbb{C}$.

From now on, given any set $U \subset \mathbb{C}$, we denote and define:

$$U^c := \mathbb{C} \backslash U = \text{the complement of } U \text{ in } \mathbb{C}$$
$$U^\circ := \text{set of all interior points of } U$$
$$\partial U := \text{set of all boundary points of } U$$
$$\overline{U} := U \cup \partial U = \text{the closure of } U$$

There is no standard notation for the set of all exterior points though. According to the definition of interior points, we must have $U^\circ \subset U$.

We are now ready to define what are open sets and closed sets. The way we define open sets is very common in many other textbooks, while the way we define closed sets may sound different from some textbooks but it is more intuitive and is nonetheless equivalent to the definition found in other textbooks.

Definition 1.33 (Open and Closed Sets)**.** A set $\Omega \subset \mathbb{C}$ is said to be *open* if every point $z \in \Omega$ is an interior point of Ω (i.e. $\Omega = \Omega^\circ$). A set $E \subset \mathbb{C}$ is said to be *closed* if all boundary points of E belong to E (i.e. $\partial E \subset E$).

Remark 1.34. Note that it is always true that $\Omega^\circ \subset \Omega$.

Let's look at some examples. Consider the set $\Omega = \{z \in \mathbb{C} : \text{Re}(z) > 1\}$:

$$\Omega^\circ = \{z \in \mathbb{C} : \text{Re}(z) > 1\} = \Omega$$
$$\partial\Omega = \{z \in \mathbb{C} : \text{Re}(z) = 1\} \not\subset \Omega$$

Therefore, Ω is an open set, but is not closed.

Let's look at another example: $E = \{z \in \mathbb{C} : \text{Re}(z) \geq 1\}$. By inspection (we left the detail for readers), we can see:

$$E^\circ = \{z \in \mathbb{C} : \text{Re}(z) > 1\} \neq E$$
$$\partial E = \{z \in \mathbb{C} : \text{Re}(z) = 1\} \subset E$$

Therefore, E is not open, but is closed.

There are sets which are not open and not closed! For instance, consider the unit circle $W = \{z \in \mathbb{C} : \text{Re}(z) \geq 0 \text{ and } \text{Im}(z) > 0\}$. We can see from Figure 1.3 that:

$$W^\circ = \{z \in \mathbb{C} : \text{Re}(z) > 0 \text{ and } \text{Im}(z) > 0\} \neq W$$
$$\partial W = \{x + 0i \in \mathbb{C} : x \geq 0\} \cup \{0 + yi \in \mathbb{C} : y \geq 0\} \not\subset W.$$

W is neither open nor closed.

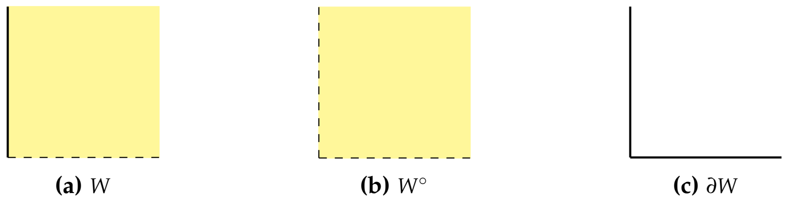


Figure 1.3. The set $W = \{z \in \mathbb{C} : \mathrm{Re}(z) \geq 0 \text{ and } \mathrm{Im}(z) > 0\}$ and its interior and boundary sets

Surprisingly, there are sets which are *both* open and closed (so "open" and "closed" are not exactly opposite, which is a linguistic nightmare)! For subsets of $\mathbb{C}$, there are not many though. They are the empty set $\varnothing$ and the whole $\mathbb{C}$. It is easy to see that $\mathbb{C}^\circ = \mathbb{C}$ and $\partial\mathbb{C} = \varnothing \subset \mathbb{C}$ (the empty-set is a subset of every set). This shows $\mathbb{C}$ is both open and closed.

The argument that shows $\varnothing$ is both open and closed has a bit of *philosophical* favor. We claim that $\varnothing^\circ = \varnothing$. Suppose otherwise, then we must have $\varnothing^\circ \not\subset \varnothing$ (since $\varnothing$ is a subset of every set). This means there exists $z \in \varnothing^\circ$ such that $z \notin \varnothing$. Then, z being an interior point of $\varnothing$ implies there exists $\varepsilon > 0$ such that $B_\varepsilon(z) \subset \varnothing$, which is clearly impossible! This shows $\varnothing^\circ = \varnothing$ and so the empty set is open. To show $\varnothing$ is closed as well, we claim $\partial\varnothing = \varnothing$. Suppose $\partial\varnothing$ is non-empty, then we can pick $w \in \partial\varnothing$, then for any $\delta > 0$, both $B_\delta(w) \cap \varnothing$ and $B_\delta(w) \cap (\mathbb{C} \backslash \varnothing)$ are non-empty. However, the former cannot happen! This concludes $\partial\varnothing = \varnothing$, and so $\varnothing$ is closed as well!

Remark 1.35. There is an interesting YouTube video titled "Hitler learns Topology".

Exercise 1.25. Determine whether each set U_1 to U_5 in Exercise 1.24 is open, closed, neither or both.

Readers who have learned a bit point-set topology may have seen another definition of closed sets, namely a set E is closed if its complement E^c is open. We are going to show that this is equivalent to our definition:

Proposition 1.36. *For any set $E \subset \mathbb{C}$, we have*

$$\partial E \subset E \quad \Longleftrightarrow \quad E^c \text{ is open.}$$

Proof. ($\Longrightarrow$)-part: Suppose $\partial E \subset E$. Consider $z \in E^c$, by the given condition $\partial E \subset E$, we know $z \notin \partial E$. This shows there exists $\varepsilon > 0$ such that at least one of the sets $B_\varepsilon(z) \cap E$ or $B_\varepsilon(z) \cap E^c$ is empty. Since $z \in E^c$, we must have $B_\varepsilon(z) \cap E = \varnothing$, which is equivalent to saying $B_\varepsilon(z) \subset E^c$. This shows E^c is open.

($\Longleftarrow$)-part: Suppose E^c is open. Consider $w \in \partial E$, and we need to show $w \in E$. Suppose not, then $w \in E^c$. By the openness of E^c, there exists $\delta > 0$ such that $B_\delta(w) \subset E^c$. However, it would imply $B_\delta(w) \cap E = \varnothing$, contradicting to the fact that $w \in \partial E$. This shows $w \in E$, completing the proof that $\partial E \subset E$. □

Therefore, from now on we can say a set is closed if and only if its complement is open, which is more convenient sometimes. For instance, this fact can be used to show an important and nice property about a closed set E: if there is a *convergent* sequence in E, then the limit must be inside E.

Proposition 1.37. *Let $E \subset \mathbb{C}$ be a closed set. Suppose $\{z_n\}_{n=1}^\infty$ is a complex sequence such that $z_n \in E$ for any n. If $\lim\limits_{n\to\infty} z_n = w$, then $w \in E$.*

Proof. We prove by contradiction. The key idea is that if $w \notin E$, then one can draw a small ball around w such that the ball is completely outside E. However, then z_n which approaches w must go within the ball, and hence outside E, when n is large (see Figure 1.4).

Here we present the detail: suppose $w \notin E$, then $w \in E^c$. By Proposition 1.36, E^c is open and so there exists $\varepsilon > 0$ such that $B_\varepsilon(w) \subset E^c$. By the fact that $z_n \to w$, there exists $N \in \mathbb{N}$ such that whenever $n \geq N$, we have $|z_n - w| < \varepsilon$. However, it implies:

$$z_n \in B_\varepsilon(w) \subset E^c \quad \Longrightarrow \quad z_n \notin E$$

which is clearly a contradiction. It proves $w \in E$. □

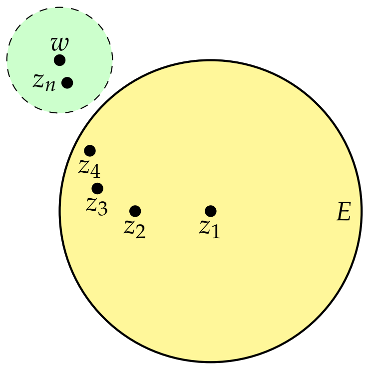


Figure 1.4. If E is closed, $w \notin E$ and $z_n \to w$, then z_n must go outside E for large n.

Below is a list of useful facts about open and closed sets. We will prove some of them and leave the others as exercises for readers.

Proposition 1.38. *Open and closed sets in $\mathbb{C}$ have the following properties:*

- *The union $\bigcup_\alpha U_\alpha$ of any family (finite or infinite) of open sets $\{U_\alpha\}$ in $\mathbb{C}$ is open.*
- *The intersection $\bigcap_{k=1}^{N} U_k$ of a finite family of open sets $U_1, \ldots, U_N$ in $\mathbb{C}$ is open.*
- *The union $\bigcup_{k=1}^{N} E_k$ of a finite family of closed sets $E_1, \ldots, E_N$ in $\mathbb{C}$ is closed.*
- *The intersection $\bigcap_\alpha E_\alpha$ of any family (finite or infinite) of closed sets $\{E_\alpha\}$ in $\mathbb{C}$ is closed.*

Proof. Let's prove the second statement only, that if $U_1, \ldots, U_N$ are open, then their intersection is also open. Let $z \in \bigcap_{k=1}^{N} U_k$, then $z \in U_k$ for any $k = 1, \ldots, N$. For each k, since U_k is open, z is an interior point of U_k and so there exists $\varepsilon_k > 0$ such that $B_{\varepsilon_k}(z) \subset U_k$. Let $\varepsilon = \min\{\varepsilon_1, \ldots, \varepsilon_N\}$, which is positive, then $\varepsilon \leq \varepsilon_k$ for any k, and so we have:

$$B_\varepsilon(z) \subset B_{\varepsilon_k}(z) \subset U_k \text{ for any } k = 1, \ldots, N.$$

Therefore, $B_\varepsilon(z) \subset \bigcap_{k=1}^{N} U_k$. This shows z is an interior point of $\bigcap_{k=1}^{N} U_k$. As a result, $\bigcap_{k=1}^{N} U_k$ is an open set.

We leave the proof of the first statement as an exercise for readers. Once the first two statements are established, the third and fourth statements about closed sets easily follow from Proposition 1.36 and De Morgan's Rule: $\left(\bigcup_k E_k\right)^c = \bigcap_k E_k^c$ and $\left(\bigcap_\alpha E_\alpha\right)^c = \bigcup_\alpha E_\alpha^c$. □

Exercise 1.26. Prove all the other three statements in Proposition 1.38. Give an example of a family of open sets whose intersection is not open. Also give an example of a family of closed sets whose union is not closed.

Exercise 1.27. Given any two sets U and V in $\mathbb{C}$, show that:

(a) $\partial(U \cup V) \subset \partial U \cup \partial V$

(b) $\partial(\partial U) = \partial U$

(c) $\overline{U} := U \cup \partial U$ is always closed.

Here are two more terminologies which we will use sometimes:

- A set Ω in $\mathbb{C}$ is said to be *bounded* if there exists $M > 0$ such that $|z| < M$ for any $z \in \Omega$, i.e. $\Omega \subset B_M(0)$.
- A set Ω in $\mathbb{C}$ is said to be *compact* if it is *closed* and *bounded*.

Exercise 1.28. Use the Bolzano-Weierstrass's Theorem for $\mathbb{R}$ to show the Bolzano-Weierstrass's Theorem for $\mathbb{C}$, which asserts that if $\{z_n\}_{n=1}^{\infty}$ is a complex sequence in a bounded set Ω, then there exists a convergent subsequence $\{z_{n_k}\}_{k=1}^{\infty}$ of $\{z_n\}_{n=1}^{\infty}$.

Exercise 1.29. Supoose $\Omega_1 \supset \Omega_2 \supset \Omega_3 \supset \cdots$ is an infinite sequence of non-empty *compact* sets in $\mathbb{C}$. Show that:

$$\bigcap_{k=1}^{\infty} \Omega_k \neq \emptyset.$$

[*Hint: Pick $z_k \in \Omega_k$ for each k. What can you say about $\{z_k\}_{k=1}^{\infty}$?*]

1.3.2. Connected Sets. Intuitively, a connected set is one that is in one "piece". However, such a definition is not rigorous as the word "piece" is quite vague. To define connectedness, we first need to understand what it means by a *disconnected set*:

Definition 1.39 (Disconnected Sets). A set $\Omega \subset \mathbb{C}$ is said to be *disconnected* if there exists two disjoint open sets U and V (disjoint means $U \cap V = \emptyset$) such that:

$$\Omega \subset U \cup V, \quad \Omega \cap U \neq \emptyset \quad \text{and} \quad \Omega \cap V \neq \emptyset.$$

Remark 1.40. The condition $\Omega \subset U \cup V$ means that U and V together cover the whole set Ω. The condition $\Omega \cap U$ and $\Omega \cap V$ being non-empty means that Ω has something inside U and something inside V. Since the definition requires U and V are disjoint (i.e. separated in some sense), these sets U and V create a separation for the set Ω, and hence we say Ω is disconnected (see Figure 1.5).

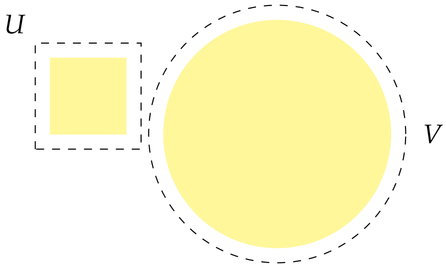
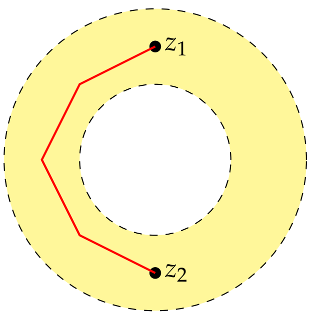


Figure 1.5. Ω is the yellow set. It is disconnected with disjoint open sets U and V that separate Ω.

A set Ω is said to be *connected* if it is *not disconnected*, meaning that whenever there are disjoint open sets U and V covering the set Ω, then at least one of $\Omega \cap U$ or $\Omega \cap V$ must be empty. In practice, it is not straight-forward to verify that a set is connected using the definition, even for simple examples such as an open ball $B_r(z)$, an open rectangle or an annulus $1 < |z| < 2$. However, thanks for a proposition that we will state, one can verify that they are all connected easily. Before we state the proposition, we need to define:

Definition 1.41 (Polygonally Path-Connected Sets)**.** A non-empty set $\Omega \subset \mathbb{C}$ is said to be *polygonally path-connected* if any pair of points in Ω can be joined by a continuous path consisting of finitely many line segments contained inside Ω.

For instance, any convex set is polygonally path-connected since every pair of points can be joined by a single line segment contained inside the set. The annulus $1 < |z| < 2$ is also polygonally path-connected (see the figure below):



The following proposition asserts that for any *open* set Ω, *connectedness* and *polygonal-path-connectedness* are equivalent:

Proposition 1.42. *An open set Ω in $\mathbb{C}$ is connected if and only if it is polygonally path-connected.*

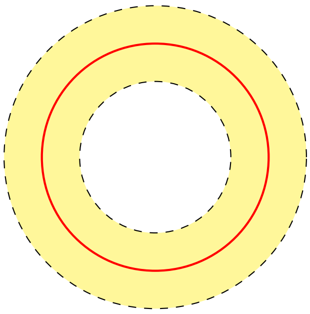
We omit the proof in this lecture note. Interested readers may consult Stein-Shakarchi's book (Exercise 5 in P.25) for an outline of the proof and try to complete the detail as an exercise. Using this proposition, it is easy to see that any convex open sets (and many other non-convex open sets) are connected.

The last notion about sets in $\mathbb{C}$ to be introduced is *simply-connectedness*. Readers should have encountered this concept in Multivariable Calculus (typically in the chapter about conservative vector field).

Definition 1.43 (Simply-Connected Sets)**.** A set Ω is said to be *simply-connected* if Ω is connected and that every closed loop in Ω can continuously contract to a point without leaving Ω.

The concept of simply-connectedness will come up frequently when we talk contour integrals and Cauchy's Integral Formula.

A ball and a rectangle (either open or closed) are simply-connected, while an annulus $1 < |z| < 2$ is not, because the red circle in the figure below cannot shrink to a point unless it steps into the "hole" which is not a part of the annulus.



On $\mathbb{C}$, simply-connected sets have one nice property concerning simple closed curves ("simple closed" means closed without self-intersections). If γ is a simple closed curve contained inside a simply-connected set Ω, then the region enclosed by γ will be a subset of Ω. Some textbooks put this as the definition of simply-connected sets in $\mathbb{C}$.

Exercise 1.30. For each set described below, sketch the region on $\mathbb{C}$, and determine whether it is (i) open, (ii) closed, (iii) bounded, (iv) compact, (v) connected and (vi) simply-connected or not.

(a) $\Omega_1 = \{z \in \mathbb{C} : |z+1| \geq 4\,|z-1|\}$

(b) $\Omega_2 = \{z \in \mathbb{C} : |z+1| < 4\,|z-1|\}$

(c) $\Omega_3 = \{z \in \mathbb{C} : |z| \leq \mathrm{Re}(z) + 1\}$

(d) $\Omega_4 = \{e^z \in \mathbb{C} : 1 \leq \mathrm{Re}(z) \leq 2\}$

(e) $\Omega_5 = \{z \in \mathbb{C} : |z^2 - 1| \leq 1\}$