Chapter 5

What is the Riemann Hypothesis?

5.1. Analytic Continuation

We will end this course by an introduction to the Riemann Hypothesis, a long-standing unresolved problem in Pure Mathematics, and is a topic of central importance in Complex Analysis, Number Theory, and related fields.

The Riemann Hypothesis concerns about the Riemann zeta function which is *a priori* defined by the following infinite sum:

$$\zeta(z) := \sum_{n=1}^{\infty} \frac{1}{n^z} = \sum_{n=1}^{\infty} \frac{1}{e^{z \ln n}}.$$

Here n^z is regarded as a single-valued function of z. This sum converges absolutely on the domain $\Omega = \{z \in \mathbb{C} : \mathrm{Re}(z) > 1\}$, and converges uniformly on every smaller domain $\Omega_\varepsilon = \{z \in \mathbb{C} : \mathrm{Re}(z) > 1 + \varepsilon\}$. Therefore, Morera's Theorem shows that ζ is holomorphic on Ω.

Although ζ is a priori defined on Ω, we will soon learn that it can be extended to a holomorphic function on $\mathbb{C}\backslash\{1\}$. In other words, there exists a function $\hat{\zeta} : \mathbb{C}\backslash\{1\} \to \mathbb{C}$ such that $\hat{\zeta}(z) = \zeta(z)$ for any $z \in \Omega$, and that $\hat{\zeta}$ is holomorphic on $\mathbb{C}\backslash\{1\}$. This new function $\hat{\zeta}$ is called the *analytic continuation* of ζ.

Such an analytic continuation can be shown to be unique, and it is common to abuse the notations a bit by simply writing ζ (instead of $\hat{\zeta}$) for the analytic continuation of ζ. In this section, we will collect some useful facts about analytic continuations. We will then describe how to extend ζ in the next section.

Definition 5.1 (Analytic Continuations)**.** Given a holomorphic function $f : \Omega \to \mathbb{C}$, a function $\hat{f} : \hat{\Omega} \to \mathbb{C}$ defined on a connected domain $\hat{\Omega} \supset \Omega$ is said to be an *analytic continuation* of f on $\hat{\Omega}$ if:

- $\hat{f}(z) = f(z)$ for any $z \in \Omega$; and
- $\hat{f}$ is holomorphic on $\hat{\Omega}$.

While a (real) differentiable function defined on a smaller domain can be easily extended to a (real) differentiable function defined on a larger domain, it is very difficult to do so for a holomorphic function. One reason is that holomorphic functions are very *rigid*, in a sense that if any two holomorphic functions coincide on an open set, then the two function must be equal elsewhere! As a corollary, if an analytic continuation exists, then it must be unique! Let's state and prove this fact:

Theorem 5.2 (Identity Theorem). *Let $f : \Omega \to \mathbb{C}$ be a holomorphic function on a connected domain Ω. If there exists a non-empty open set $U \subset \Omega$ such that $f(z) = 0$ for any $z \in U$, then $f \equiv 0$ on Ω.*

Proof. Consider the set

$$S := \{z \in \Omega : f^{(n)}(z) = 0 \text{ for any } n \geq 0\}.$$

Since $f(z) = 0$ on U which is an open set, we have $f(z) = f'(z) = f''(z) = \cdots = 0$ for any $z \in U$. This shows $U \subset S$, and so S is non-empty. The proof goes by showing S is both closed and open. Together with the fact that S is non-empty and Ω is connected, it will prove $S = \Omega$ which implies our claim.

To show S is closed, we recall the fact that a holomorphic function f must be infinitely differentiable, and hence $f^{(n)}$ are all continuous functions. The set S can be written as:

$$S := \bigcap_{n=0}^{\infty} \left(f^{(n)}\right)^{-1}(0).$$

The single set $\{0\}$ is closed, and hence the pre-image $\left(f^{(n)}\right)^{-1}(0)$ is closed for each $n \geq 0$. Since the intersection of any family of closed sets is closed, we conclude that S is closed.

To show S is open, we consider Taylor series expansions. For any $z_0 \in \Omega$, we consider the Taylor series about z_0 of f:

$$f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!}(z - z_0)^n$$

which is defined on an open ball $B_\varepsilon(z_0)$ for some $\varepsilon > 0$ (according to Taylor's Theorem). If $z_0 \in S$, then we will have $f^{(n)}(z_0) = 0$ for any $n \geq 0$, and as such, the above Taylor series shows $f(z) = 0$ for any $z \in B_\varepsilon(z_0)$. In other words, $B_\varepsilon(z_0) \subset S$. This shows S is open.

Finally, S is non-empty, open and closed, and Ω is connected, so $S = \Omega$. $\square$

Corollary 5.3. *Suppose $g : \Omega \to \mathbb{C}$ and $h : \Omega \to \mathbb{C}$ are two holomorphic functions defined on a connected domain Ω, and that g and h coincide on a smaller open set $U \subset \Omega$, then it is necessary that $g \equiv h$ on Ω.*

Proof. Apply $f := g - h$ to Identity Theorem. $\square$

As a result, an analytic continuation $\hat{f}$ of a holomorphic function f, if exists, must be unique. It makes it very difficult to find such an extension!

Exercise 5.1. Why is it necessary for f to be holomorphic in the proof of Identity Theorem? Point out which part of the proof is no longer valid if f is just assumed to smooth (differentiable for infinitely many times).

Example 5.1. Consider the series:

$$f(z) = \sum_{n=0}^{\infty} z^n$$

which converges pointwise on $B_1(0)$, and uniformly on every smaller ball $B_{1-\varepsilon}(0)$ where $\varepsilon > 0$. Therefore, $f : B_1(0) \to \mathbb{C}$ is a holomorphic function on $B_1(0)$.

On the other hand, the infinite sum is:

$$\sum_{n=0}^{\infty} z^n = \frac{1}{1-z},$$

and the function $\hat{f}(z) = \dfrac{1}{1-z}$ is defined on every $z \in \mathbb{C}\backslash\{1\}$, not only those in $B_1(0)$. Therefore, $\hat{f} : \mathbb{C}\backslash\{1\} \to \mathbb{C}$ is the analytic continuation of f on $\mathbb{C}\backslash\{1\}$.

Exercise 5.2. What's wrong with the following claim?

From $\hat{f}(-1) = f(-1)$ (where f and $\hat{f}$ are defined as in Example 5.1), we have:

$$\sum_{n=0}^{\infty} (-1)^n = \frac{1}{1-(-1)} = \frac{1}{2}.$$

Hence:

$$1 - 1 + 1 - 1 + 1 - 1 + \cdots = \frac{1}{2}.$$

Exercise 5.3. Consider the following function defined by the sum:

$$f(z) = 1 + \frac{1}{z} + \frac{1}{z^2} + \cdots = \sum_{n=0}^{\infty} \frac{1}{z^n}.$$

What is the largest possible domain on which f is holomorphic? Find the analytic continuation of f on the larger domain $\mathbb{C}\backslash\{1\}$. Is it possible to further extend the function to become an entire function on $\mathbb{C}$?

Another common way of extending a holomorphic function is through a *functional equation*. Let's consider the following example. Suppose $f : \Omega \to \mathbb{C}$ is a holomorphic function on $\Omega := \{z \in \mathbb{C} : \text{Re}(z) > 1\}$. If it can be shown that f satisfies an equation such as:

$$f(z+1) = 2f(z) \quad \text{for any } z \in \Omega,$$

then one can define an analytic continuation of it by the following way:

$$\hat{f}(z) := \frac{1}{2} f(z+1).$$

Since $f(z+1)$ is well-defined as long as $z+1 \in \Omega$, or equivalently, $\text{Re}(z) > 0$, the extend function $\hat{f}(z)$ is now defined on a larger domain $\hat{\Omega} := \{z \in \mathbb{C} : \text{Re}(z) > 0\}$. Note that $\hat{f}(z) = \frac{1}{2} f(z+1)$ is holomorphic on $\{\text{Re}(z) > 0\}$ since f is so on $\{\text{Re}(z) > 1\}$. Also, when $\text{Re}(z) > 1$, we have

$$\hat{f}(z) = \frac{1}{2} f(z+1) = f(z)$$

by the given functional equation. Therefore, $\hat{f}$ is the analytic continuation of f on $\{\text{Re}(z) > 0\}$.

Furthermore, the same functional equation holds for $\hat{f}$. Let's verify this. For any z such that $\mathrm{Re}(z) > 0$, we have:

$$\begin{aligned}\hat{f}(z+1) - 2\hat{f}(z) &= \frac{1}{2}f(z+2) - 2\cdot\frac{1}{2}f(z+1)\\ &= \frac{1}{2}\left(f(z+2) - 2f(z+1)\right) = 0.\end{aligned}$$

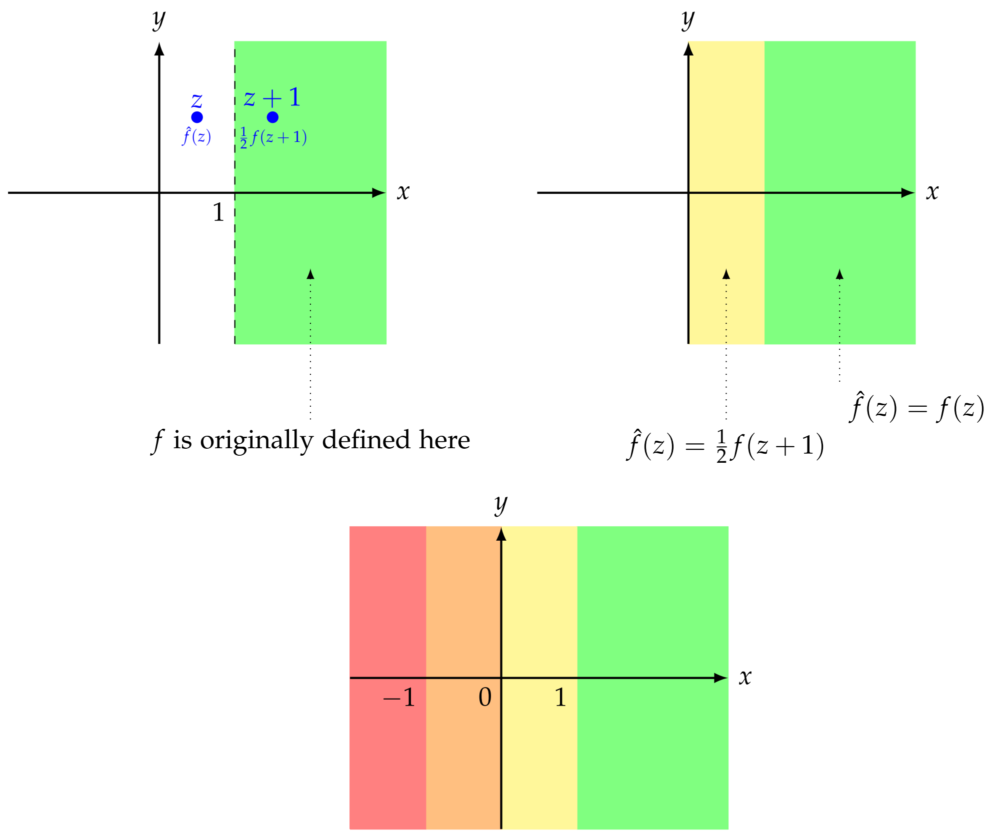
Now that $\hat{f}$ is holomorphic on $\{z : \mathrm{Re}(z) > 0\}$ and satisfies the functional equation

$$\hat{f}(z+1) = 2\hat{f}(z).$$

One can then repeat the same procedure as before to extend $\hat{f}$ to a holomorphic function $\hat{\hat{f}}$ defined on $\{z : \mathrm{Re}(z) > -1\}$, which is given by:

$$\hat{\hat{f}}(z) = \frac{1}{2}\hat{f}(z+1), \quad \text{for any } z \in \{\mathrm{Re}(z) > -1\}.$$

Inductively, we can repeat the same procedure over and over again, and extend f to a function $F : \mathbb{C} \to \mathbb{C}$ that is holomorphic on the whole complex plane $\mathbb{C}$.



f can be inductively extended to an entire function F

Exercise 5.4. Given that $f : \Omega \to \mathbb{C}$ is a holomorphic on $\Omega := \{z : \mathrm{Re}(z) > 1\}$, and that it satisfies the relation $f(z+1) = zf(z)$ for any $z \in \Omega$. Show that there is an analytic continuation $\hat{f}$ on $\mathbb{C}\backslash\{0, -1, -2, -3, \cdots\}$. Classify the type of singularities (pole, removable or essential singularity) of each non-positive integer $-n$ for $\hat{f}$.

5.2. Riemann ζ Functions

5.2.1. Analytic Continuation of Γ. In this section we discuss the Γ (Gamma) and ζ (zeta) functions, as well as their analytic continuations. These two functions are closely related. The Gamma function $\Gamma : \Omega \to \mathbb{C}$ is *a priori* defined on $\Omega := \{z : \mathrm{Re}(z) > 0\}$ by:

$$\Gamma(z) := \int_0^\infty t^{z-1}e^{-t}\,dt \quad \text{for } \mathrm{Re}(z) > 0.$$

It is an improper integral. By breaking it down into:

$$\Gamma(z) = \int_0^1 t^{z-1}e^{-t}\,dt + \int_1^\infty t^{z-1}e^{-t}\,dt,$$

one can verify (as an exercise) that the first integral is integrable when $\mathrm{Re}(z) > 0$, and the second integral is integrable for any $z \in \mathbb{C}$.

Exercise 5.5. Show that:

(a) $\displaystyle\int_0^1 t^{z-1}e^{-t}\,dt$ is integrable when $\mathrm{Re}(z) > 0$; and

(b) $\displaystyle\int_c^\infty t^{z-1}e^{-t}\,dt$ is integrable for any $z \in \mathbb{C}$ for any $c > 0$.

Exercise 5.6. Use Morera's Theorem to show that Γ is holomorphic on $\{\mathrm{Re}(z) > 0\}$. Hint: Note that t^{z-1} is holomorphic for each fixed $t > 0$, but not when $t = 0$. Morera's Theorem cannot be directly applied on this integral. To tackle this issue, consider the sequence of functions:

$$f_n(z) := \int_{\frac{1}{n}}^\infty t^{z-1}e^{-t}\,dt.$$

Show that f_n is holomorphic on $\{\mathrm{Re}(z) > 0\}$ for each n, and that f_n converges uniformly to Γ on $\{\mathrm{Re}(z) > 0\}$ as $n \to \infty$.

Using integration-by-parts, one can derive a functional equation for Γ which can be used to extend Γ beyond the domain $\{\mathrm{Re}(z) > 0\}$. For any $\mathrm{Re}(z) > 0$, we consider:

$$\begin{aligned}
\Gamma(z+1) &= \int_0^\infty t^z e^{-t}\,dt = \int_0^\infty t^z\,d(-e^{-t}) \\
&= \left[-t^z e^{-t}\right]_{t=0}^{t=\infty} + \int_0^\infty e^{-t} d(t^z) \\
&= 0 + \int_0^\infty z t^{z-1} e^{-t}\,dt \\
&= z\Gamma(z).
\end{aligned}$$

We leave the part $\left[-t^z e^{-t}\right]_{t=0}^{t=\infty} = 0$ as an exercise for readers:

Exercise 5.7. Show that whenever $\mathrm{Re}(z) > 0$, we have:

$$\lim_{t\to 0^+} t^z e^{-t} = 0 \quad \text{and} \quad \lim_{t\to\infty} t^z e^{-t} = 0.$$

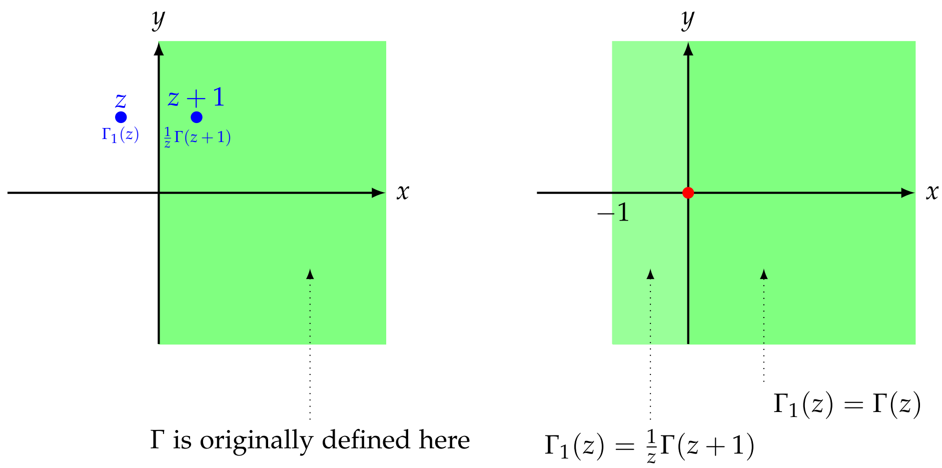
Exercise 5.8. Show that for any positive integer n, we have:

$$\Gamma(n) = (n-1)!$$

From the functional equation $\Gamma(z+1) = z\Gamma(z)$, one can define:

$$\Gamma_1(z) := \frac{1}{z}\Gamma(z+1)$$

for any z such that $z \neq 0$ and $\mathrm{Re}(z+1) > 0$. Then, Γ_1 is an holomorphic function on $\{z : \mathrm{Re}(z) > -1\}\backslash\{0\}$, and when $\mathrm{Re}(z) > 0$, we have $\Gamma_1(z) = \Gamma(z)$. In other words, Γ_1 is an analytic continuation of Γ.



The functional equation for Γ then induces a new functional equation for Γ_1. Whenever $\mathrm{Re}(z) > -1$, we have:

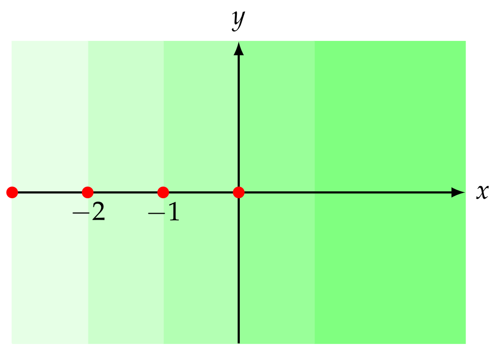
$$\begin{aligned}
\Gamma_1(z+1) &= \frac{1}{z+1}\Gamma(z+2) && \text{(Definition of } \Gamma_1\text{)}\\
&= \frac{1}{z+1}\cdot(z+1)\Gamma(z+1) && \text{(Functional equation for } \Gamma\text{)}\\
&= \Gamma(z+1) = z\Gamma_1(z) && \text{(Definition of } \Gamma_1\text{).}
\end{aligned}$$

Therefore, one can define:

$$\Gamma_2(z) := \frac{1}{z}\Gamma_1(z+1)$$

for any $z \in \mathbb{C}$ such that $z+1$ is in the domain of Γ_1, i.e. $\mathrm{Re}(z) > -2$ and $z \neq -1$. As such, Γ_2 is an analytic continuation of Γ_1 (and hence of Γ) on $\{\mathrm{Re}(z) > -2\}\backslash\{0, -1\}$.

Repeat the above process indefinitely, one can define analytic continuations Γ_m on $\{\mathrm{Re}(z) > -m\}\backslash\{0, -1, -2, \cdots, -(m-1)\}$, and eventually an analytic continuation $\hat{\Gamma}$ of Γ on the domain $\mathbb{C}\backslash\{0, -1, -2, -3, \cdots\}$.



Γ can be inductively extended to $\hat{\Gamma}$

Exercise 5.9. Show that for any integer $m \geq 1$ and z in the domain of Γ_m, we have:

$$\Gamma_m(z) = \frac{\Gamma(z+m)}{z(z+1)\cdots(z+m-1)}$$

Exercise 5.10. Show that each non-positive integer $-n$ is a simple pole of $\hat{\Gamma}$, and that:

$$\mathrm{Res}(\hat{\Gamma}, -n) = \frac{(-1)^n}{n!}.$$

Here is a summary of facts about the Gamma function:

- Γ is a priori defined on $\{z : \mathrm{Re}(z) > 0\}$.
- By the relation $\Gamma(z+1) = z\Gamma(z)$, one can define an analytic continuation $\hat{\Gamma}$ of Γ on $\mathbb{C}\backslash\{0, -1, -2, -3, \cdots\}$.
- Each non-positive integer $-n$ is a simple pole of $\hat{\Gamma}$, with residue equal to $\frac{(-1)^n}{n!}$.

Recall that since the analytic continuation must be unique, some textbooks denote the analytic continuation by simply Γ.

Exercise 5.11. Show that when $\mathrm{Re}(z) > 0$, the Gamma function can be decomposed into:

$$\Gamma(z) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!(z+n)} + \int_1^{\infty} e^{-t} t^{z-1}\, dt.$$

Show also that the infinite sum $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n}{n!(z+n)}$ converges for any $z \neq 0, -1, -2, -3, \cdots$ and the integral $\displaystyle\int_1^{\infty} e^{-t} t^{z-1}\, dt$ is an entire function of z.

5.2.2. Relation between Γ and ζ.

Recall that the Riemann zeta function $\zeta : \{z : \mathrm{Re}(z) > 1\} \to \mathbb{C}$ is defined by the infinite series:

$$\zeta(z) = \sum_{n=1}^{\infty} \frac{1}{n^z}$$

which converges when $\mathrm{Re}(z) > 1$. The following lemma shows a relation between Γ and ζ.

Lemma 5.4. *For any $z \in \mathbb{C}$ such that $\mathrm{Re}(z) > 1$, we have:*

$$\zeta(z)\Gamma(z) = \int_0^{\infty} \frac{t^{z-1}}{e^t - 1}\, dt. \tag{5.1}$$

Proof. The key step of the proof is the change of variables $t = n\tau$ in the integral that defines Γ:

$$\begin{aligned}
\Gamma(z) &= \int_0^{\infty} t^{z-1} e^{-t}\, dt = \int_0^{\infty} (n\tau)^{z-1} e^{-n\tau}\, d(n\tau) \\
&= n^z \int_0^{\infty} \tau^{z-1} e^{-n\tau}\, d\tau \\
\frac{1}{n^z}\Gamma(z) &= \int_0^{\infty} t^{z-1} e^{-nt}\, dt.
\end{aligned}$$

Here we have used the fact that τ is a dummy variable. Summing up over n, we get:

$$\sum_{n=1}^{\infty} \frac{1}{n^z}\Gamma(z) = \sum_{n=1}^{\infty} \int_0^{\infty} t^{z-1}e^{-nt}\,dt. \tag{5.2}$$

Next we want to switch the integral and summation signs. It has to be justified using LDCT. Consider:

$$\left|t^{z-1}e^{-nt}\right| \leq t^{x-1}e^{-nt}$$

for any $t \in [0, \infty)$. Note that:

$$\sum_{n=1}^{\infty} \int_0^{\infty} t^{x-1}e^{-nt}\,dt = \sum_{n=1}^{\infty} \frac{1}{n^x}\Gamma(x)$$

which converges since $x > 1$. Hence, LDCT shows we can switch the summation and integral signs of (5.2), and it yields:

$$\sum_{n=1}^{\infty} \frac{1}{n^z}\Gamma(z) = \int_0^{\infty} t^{z-1} \sum_{n=1}^{\infty} e^{-nt}\,dt.$$

Observing that $\displaystyle\sum_{n=1}^{\infty} e^{-nt} = \sum_{n=1}^{\infty} \left(e^{-t}\right)^n$ is a geometric series, we get:

$$\sum_{n=1}^{\infty} e^{-nt} = \frac{e^{-t}}{1-e^{-t}} = \frac{1}{e^t - 1}.$$

From (5.2), we get our desired result (5.1). □

5.2.3. Analytic Continuation of ζ. The relation (5.1) will be used to extend ζ beyond the domain $\{\mathrm{Re}(z) > 1\}$. We have already shown that Γ can be extended to almost all of $\mathbb{C}$. If we are able to extend the integral:

$$\int_0^{\infty} \frac{t^{z-1}}{e^t - 1}\,dt$$

beyond $\{\mathrm{Re}(z) > 1\}$, then ζ can also be extended accordingly.

First break down the integral into two part:

$$\int_0^{\infty} \frac{t^{z-1}}{e^t - 1}\,dt = \int_0^{1} \frac{t^{z-1}}{e^t - 1}\,dt + \int_1^{\infty} \frac{t^{z-1}}{e^t - 1}\,dt.$$

The second integral is well-defined for any $z \in \mathbb{C}$. To see this, we first note that $\left|t^{z-1}\right| = t^{x-1} \ll e^{t/2}$ as $t \to \infty$, and hence $\dfrac{t^{x-1}}{e^t - 1} \ll \dfrac{e^{t/2}}{e^t - 1} \sim e^{-t/2}$. The function $e^{-t/2}$ is integrable over $[1, \infty)$. By comparison, the integral $\displaystyle\int_1^{\infty} \left|\frac{t^{z-1}}{e^t - 1}\right| dt$ is finite for any $z \in \mathbb{C}$ (not only those with $\mathrm{Re}(z) > 1$). By Morera's Theorem, the integral is an entire function of z.

Next we handle the first integral $\displaystyle\int_0^{1} \frac{t^{z-1}}{e^t - 1}\,dt$. The key trick is to consider the denominator $\dfrac{1}{e^t - 1}$, and expand it as a series. Consider the function:

$$f(w) = \frac{1}{e^w - 1} - \frac{1}{w}.$$

Although it is not defined when $w = 0$, we can see that 0 is a removable singularity:

$$\begin{aligned}\lim_{w\to 0} f(w) &= \lim_{w\to 0}\left(\frac{1}{w+\frac{w^2}{2!}+\frac{w^3}{3!}+\cdots}-\frac{1}{w}\right)\\ &= \lim_{w\to 0}\frac{w-w-\frac{w^2}{2!}-\frac{w^3}{3!}-\cdots}{w(w+\frac{w^2}{2!}+\frac{w^3}{3!}+\cdots)}\\ &= \lim_{w\to 0}\frac{-\frac{1}{2}-\frac{w}{6}-\cdots}{1+\frac{w}{2}+\frac{w^2}{6}+\cdots}\\ &= -\frac{1}{2}.\end{aligned}$$

Therefore, by declaring that $f(0) = -\frac{1}{2}$, it becomes a holomorphic function defined on $B_{2\pi}(0)$ (why 2π?). Consider its Taylor series about 0:

$$\begin{aligned}f(w) &= -\frac{1}{2}+f'(0)w+\frac{f''(0)}{2!}w^2+\frac{f^{(3)}(0)}{3!}w^3+\cdots\\ \frac{1}{e^w-1}-\frac{1}{w} &= \sum_{n=0}^{\infty}\frac{f^{(n)}(0)}{n!}w^n.\end{aligned}$$

Substitute $w = t \in [0,1]$, then we get:

$$\frac{1}{e^t-1} = \frac{1}{t}+\sum_{n=0}^{\infty}\frac{f^{(n)}(0)}{n!}t^n.$$

Recall from Exercise 4.8 that the Taylor's series $\sum_{n=0}^{\infty}\frac{f^{(n)}(0)}{n!}w^n$ converges uniformly on every ball $B_{2\pi-\varepsilon}(0)$ slightly smaller than $B_{2\pi}(0)$, say $B_2(0)$. In particular, since $[0,1] \subset B_2(0)$, the convergence of the series $\sum_{n=0}^{\infty}\frac{f^{(n)}(0)}{n!}t^n$ is also uniform on $[0,1]$. When z is a fixed complex number such that $\mathrm{Re}(z) > 1$, we have $\left|t^{z-1}\right| \leq t^{x-1} \leq 1$. Therefore, the series $\sum_{n=0}^{\infty}\frac{f^{(n)}(0)}{n!}t^{n+z-1}$ also converges uniformly on $t \in [0,1]$ regarding z as fixed. Using the fact, one can write the first integral as:

$$\begin{aligned}\int_0^1\frac{t^{z-1}}{e^t-1}\,dt &= \int_0^1 t^{z-1}\left(\frac{1}{t}+\sum_{n=0}^{\infty}\frac{f^{(n)}(0)}{n!}t^n\right)dt\\ &= \int_0^1\left(t^{z-2}+\sum_{n=0}^{\infty}\frac{f^{(n)}(0)}{n!}t^{z+n-1}\right)dt\\ &= \left[\frac{t^{z-1}}{z-1}\right]_{t=0}^{t=1}+\sum_{n=0}^{\infty}\frac{f^{(n)}(0)}{n!}\left[\frac{t^{z+n}}{z+n}\right]_{t=0}^{t=1}\\ &= \frac{1}{z-1}+\sum_{n=0}^{\infty}\frac{f^{(n)}(0)}{n!}\cdot\frac{1}{z+n}.\end{aligned}$$

Here we have integrated term-by-term thanks to uniform convergence of the series.

Although the integral $\int_0^1\frac{t^{z-1}}{e^t-1}\,dt$ on the LHS is defined only when $\mathrm{Re}(z) > 1$, the RHS series is defined whenever $z \neq 1, 0, -1, -2, -3, \cdots$. Furthermore, the RHS series is holomorphic on $\Omega := \mathbb{C}\backslash\{1, 0, -1, -2, -3, \cdots\}$. To show this, it suffices to prove $\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} \cdot \frac{1}{z+n}$ converges uniformly on any small ball $B_r(z_0) \subset \Omega$. Note that the singularities $\{1, 0, -1, -2, -3, \cdots\}$ are isolated, points in $B_r(z_0)$ must be well away from the singularities. There exists $\delta > 0$ such that $|z+n| \geq \delta$ for any $z \in B_r(z_0)$ and $n = 0, 1, 2, 3, \cdots$. As a result, we have:

$$\left| \frac{f^{(n)}(0)}{n!} \cdot \frac{1}{z+n} \right| \leq \left| \frac{f^{(n)}(0)}{n!} \right| \cdot \frac{1}{\delta}.$$

By Weierstrass's M-test, the series $\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} \cdot \frac{1}{z+n}$ converges uniformly on any small ball $B_r(z_0) \subset \Omega$. By Morera's Theorem, it defines a holomorphic function on any small ball $B_r(z_0) \subset \Omega$, and so is holomorphic on Ω.

Combining the result (5.1), we have so far established that on $\{\mathrm{Re}(z) > 1\}$:

$$\zeta(z) = \frac{1}{\Gamma(z)} \left(\underbrace{\frac{1}{z-1} + \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} \cdot \frac{1}{z+n}}_{\text{extendable to } \Omega} + \underbrace{\int_1^{\infty} \frac{t^{z-1}}{e^t - 1}\, dt}_{\text{entire}} \right)$$

Since Γ has an analytic continuation $\hat{\Gamma}$ on $\mathbb{C} \backslash \{0, -1, -2, -3, \cdots\}$. From the above relation, we can then define an analytic continuation of ζ on $\mathbb{C} \backslash \{1, 0, -1, -2, -3, \cdots\}$ as:

$$\begin{aligned} \hat{\zeta}(z) &= \frac{1}{\hat{\Gamma}(z)} \left(\frac{1}{z-1} + \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} \cdot \frac{1}{z+n} + \int_1^{\infty} \frac{t^{z-1}}{e^t - 1}\, dt \right) \\ &= \frac{1}{(z-1)\hat{\Gamma}(z)} + \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} \frac{1}{(z+n)\hat{\Gamma}(z)} + \frac{1}{\hat{\Gamma}(z)} \int_1^{\infty} \frac{t^{z-1}}{e^t - 1}\, dt. \end{aligned} \tag{5.3}$$

It appears (5.3) has singularities at every $1, 0, -1, -2, -3, \cdots$, yet we can show $0, -1, -2, -3, \cdots$ are all removable. It is because $\hat{\Gamma}$ has a simple pole at every of $\{0, -1, -2, -3, \cdots\}$, so they are *zeros* of $1/\hat{\Gamma}$. Therefore, $\frac{1/\hat{\Gamma}}{z+n}$ has a removable singularity at $-n$. Precisely, for any integers $m, n \in \{0, 1, 2, 3, \cdots\}$ we have:

$$\lim_{z \to -m} \frac{1}{(z+n)\hat{\Gamma}(z)} = \begin{cases} \frac{1}{\mathrm{Res}(\hat{\Gamma}, -n)} & \text{if } m = n \\ 0 & \text{if } m \neq n \end{cases}$$

This shows $\{0, 1, 2, 3, \cdots\}$ are all removable singularities of $\hat{\zeta}$ since the following limit is finite for any $m = 0, 1, 2, 3, \cdots$

$$\lim_{z \to -m} \hat{\zeta}(z) = \frac{f^{(m)}(0)}{m!} \frac{1}{\mathrm{Res}(\hat{\Gamma}, -m).}$$

Therefore, $\zeta(z)$ can be holomorphically defined on $\mathbb{C} \backslash \{1\}$ by declaring that

$$\hat{\zeta}(-m) := \frac{f^{(m)}(0)}{m!} \frac{1}{\mathrm{Res}(\hat{\Gamma}, -m)}$$

for any $m = 0, 1, 2, \cdots$. Note that 1 is a simple pole of $\hat{\zeta}$.

5.2.4. Special Values of $\hat{\zeta}$. We will determine the value of $\hat{\zeta}$ at some special $z \in \mathbb{C}$. When $z = -m$ where $-m$ is a non-positive integer, then we have already discussed that

$$\hat{\zeta}(-m) = \frac{f^{(m)}(0)}{m!} \frac{1}{\operatorname{Res}(\hat{\Gamma}, -m)}.$$

Here f is the function:

$$f(w) = \frac{1}{e^w - 1} - \frac{1}{w}.$$

We have already figured out that $\operatorname{Res}(\hat{\Gamma}, -m) = \dfrac{(-1)^m}{m!}$, so $\hat{\zeta}(-m) = (-1)^m f^{(m)}(0)$. However, it is not straight-forward to find a general expression for $f^{(m)}(0)$, but by direct computations one can verify that the first few terms of $f^{(m)}(0)$ are given as follows:

$$f(0) = -\frac{1}{2} \qquad f'(0) = \frac{1}{12} \qquad f''(0) = 0 \qquad f^{(3)}(0) = -\frac{1}{120}.$$

Therefore, the extended Riemann zeta function $\hat{\zeta}$ takes the following values:

$$\hat{\zeta}(0) = -\frac{1}{2} \qquad \hat{\zeta}(-1) = -\frac{1}{12}$$
$$\hat{\zeta}(-2) = 0 \qquad \hat{\zeta}(-3) = \frac{1}{120}$$

To many people's surprise, the fact that $\hat{\zeta}(-1) = -\frac{1}{12}$ is used in String Theory! However, many "muggles" misunderstand the meaning of it, and misinterpret it as $\sum_{n=1}^{\infty} \frac{1}{n^{-1}} = -\frac{1}{12}$, which is mathematically wrong as $\hat{\zeta}(z) = \sum_{n=1}^{\infty} \frac{1}{n^z}$ only when $\operatorname{Re}(z) > 1$. It would lead to the following awkward and non-sense expression:

$$1 + 2 + 3 + 4 + \cdots = -\frac{1}{12}.$$

Similarly, some "amateurs" mix up $\hat{\zeta}(0) = -\frac{1}{2}$ with $\sum_{n=1}^{\infty} \frac{1}{n^0} = -\frac{1}{2}$, and $\hat{\zeta}(-2) = 0$ with $\sum_{n=1}^{\infty} \frac{1}{n^{-2}} = 0$, both would lead to awkward expressions:

$$1 + 1 + 1 + 1 + \cdots = -\frac{1}{2}.$$
$$1^2 + 2^2 + 3^2 + 4^2 + \cdots = 0.$$

5.2.5. Riemann Hypothesis. Finally, we are ready to understand the statement of the Riemann Hypothesis. It is a conjecture about the zeros of the (extended) Riemann zeta function $\hat{\zeta}$. To begin, let's first recall that for any negative integer $-m$, we have:

$$\hat{\zeta}(-m) = (-1)^m f^{(m)}(0),$$

where $f(w) = \dfrac{1}{e^w - 1} - \dfrac{1}{w}$. It is not difficult to show that $f^{(m)}(0) = 0$ for any even integer m:

Exercise 5.12. Show that

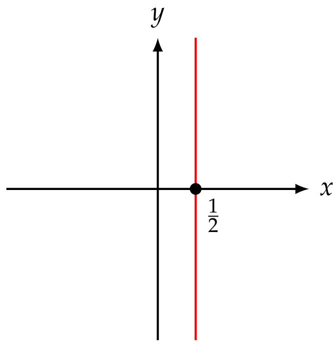
$$g(w) := f(w) + \frac{1}{2}$$

is an odd function, and hence deduce that $f^{(m)}(0) = 0$ for any even integer $m \geq 2$.

Therefore, we have $\hat{\zeta}(-2) = \hat{\zeta}(-4) = \hat{\zeta}(-6) = \cdots = 0$. These negative even integers $\{-2, -4, -6, \cdots\}$ are called *trivial zeros* of $\hat{\zeta}$.

Any complex number z_0 which is not a negative even integer is called a *non-trivial zero* of $\hat{\zeta}$ whenever $\hat{\zeta}(z_0) = 0$. The Riemann Hypothesis is concerned with the locations of these non-trivial zeros. It is conjectured by Bernhard Riemann in 1859 that:

"All non-trivial zeros z_0 of $\hat{\zeta}$ must have real part equal to $\frac{1}{2}$."



The zeros of $\hat{\zeta}$ have deep connections with the distribution of prime numbers. The renowned *Prime Number Theorem* asserts that:

$$\lim_{x\to\infty} \frac{\pi(x)}{x/\ln x} = 1$$

where $\pi(x)$ is the number of positive prime numbers less than or equal to x. A corollary of the theorem is that the n-th prime number p_n is approximately equal to $n \ln n$. The proof of Prime Number Theorem relies surprisingly on the fact that there is no zero of $\hat{\zeta}$ with real part equal to 1. If the Riemann Hypothesis is proven to be true, then the Prime Number Theorem can be substantially improved, and many mysteries about the distribution of primes will be revealed.

As of today (January 20, 2017), this conjecture remains unsolved, and is one of the most important open problem in Pure Mathematics nowadays. In 2000, the Clay Mathematics Institute compiled a list of 7 problems, called Millennium Prize Problems. For each problem in the list, the institute promises to award US$1,000,000 to the first person who solves or disproves it. Riemann Hypothesis is one of the problems in the list. The other 6 problems are: P versus NP Problem, Hodge Conjecture, Poincaré Conjecture, Yang-Mills Existence and Mass Gap, Navier-Stokes Existence and Smoothness, and Birch Swinnerton-Dyer Conjecture. The only Millennium Prize Problem that was solved is the Poincaré Conjecture, which concerns about simply-connected 3-manifolds (MATH 4033 stuff), by Grigori Perelman in 2002-03 using the idea of Ricci flow developed by Richard Hamilton in 1982.

*** End of MATH 4023 ***

**** I hope you have learned a lot and/or enjoyed the course. ****