香港科技大學
THE HONG KONG
UNIVERSITY OF SCIENCE
AND TECHNOLOGY

數學系
DEPARTMENT OF MATHEMATICS

FINAL EXAMINATION

Course Code: MATH 4023
Course Title: Complex Analysis
Semester: Spring 2016-17
Date and Time: 18 May 2017, 12:30PM - 3:30PM

Instructions

- Do **NOT** open the exam until instructed to do so.
- All mobile phones and communication devices should be switched **OFF**.
- Use of calculators is **NOT** allowed.
- It is an **OPEN-NOTES** exam. You can look at instructor's lecture notes, tutorial notes, and homework solutions. No other reference material is allowed.
- Answer **ALL** problems. Write your answers in Part A in the spaces provided, and write your solutions to problems in Part B in the yellow book.
- You must **SHOW YOUR WORK** to receive credits in all problems in Part B.
- Some problems are structured into several parts. You can quote the results stated in the preceding parts to do the next part.

HKUST Academic Honor Code

Honesty and integrity are central to the academic work of HKUST. Students of the University must observe and uphold the highest standards of academic integrity and honesty in all the work they do throughout their program of study. As members of the University community, students have the responsibility to help maintain the academic reputation of HKUST in its academic endeavors. Sanctions will be imposed on students, if they are found to have violated the regulations governing academic integrity and honesty.

"I confirm that I have answered the questions using only materials specified approved for use in this examination, that all the answers are my own work, and that I have not received any assistance during the examination."

Student's Signature: ____________________

Student's Name: ____________________ **HKUST ID:** __________

Part A - Short Questions (40 points)

1. For each f and z_0 below, determine the type of singularity of z_0. Put ✓ in the correct answer. If it is a pole, state its order. Explain briefly your answers.

   (a) $f(z) = z^{1997} + z^{1996} + \cdots + z^2 + z + 1 + \dfrac{1}{z} + \dfrac{1}{z^2} + \cdots + \dfrac{1}{z^{2047}}, \quad z_0 = 0$ /3

   ◯ pole of order _____
   ◯ essential singularity
   ◯ removable singularity

   Brief reasons:

   (b) $f(z) = \dfrac{z}{e^z - 1}, \quad z_0 = 0$ /3

   ◯ pole of order _____
   ◯ essential singularity
   ◯ removable singularity

   Brief reasons:

   (c) $f(z) = \csc z, \quad z_0 = \pi$ /3

   ◯ pole of order _____
   ◯ essential singularity
   ◯ removable singularity

   Brief reasons:

2. Is the following statement correct? /3

   "If $\mathrm{Res}(f, z_0) = 0$, then z_0 is a removal singularity of f."

   If it is correct, give a short proof of it. If not, give a counter-example.

3. Find the following residues. Explain briefly your answers.

   (a) $\text{Res}\left(\frac{1}{z}, 0\right) = \boxed{}$ /3

   Brief reasons:

   (b) $\text{Res}\left(\frac{1}{z}, 1\right) = \boxed{}$ /3

   Brief reasons:

4. Based on the proofs done in class or in the lecture notes, which of the following is/are consequence(s) of **Cauchy-Goursat's Theorem**? /4

   [Remark: If (B) is proved using (A), and (A) is a consequence of Cauchy-Goursat's Theorem, then (B) is also regarded as a consequence of Cauchy-Goursat's Theorem.]

   Put ✓ in ALL correct answer(s):

   ◯ Cauchy's integral formula
   ◯ Higher-order Cauchy's integral formula
   ◯ Liouville's Theorem
   ◯ Fundamental Theorem of Algebra
   ◯ Taylor's Theorem for holomorphic functions
   ◯ Morera's Theorem
   ◯ Residue Theorem
   ◯ Identity Theorem

5. Suppose $f_n : \mathbb{C} \to \mathbb{C}$ is a sequence of entire functions which converges to f uniformly on $B_N(0)$ for every positive integer N. Which of the following must be true? Put ✓ in **ALL** correct answer(s): /4

   ◯ f_n converges to f uniformly on $B_R(0)$ for any positive real number R.
   ◯ f_n converges to f uniformly on $\mathbb{C}$.
   ◯ f is holomorphic on $B_R(0)$ for any positive real number R.
   ◯ f is holomorphic on $\mathbb{C}$.

6. Consider a sequence of complex-valued functions $f_n : \Omega \to \mathbb{C}$ (where $n \geq 1$) defined on the domain Ω satisfying: **/4**

$$\left| \sum_{n=N+1}^{\infty} f_n(z) \right| \leq \frac{1}{N} \quad \text{and} \quad |f_n(z)| \leq \frac{1}{n} \qquad \text{for any } z \in \Omega \text{ and any } N, n \geq 1.$$

Which of the following must be true? Put ✓ in **ALL** correct answer(s):

- ○ $f_n(z)$ converges to 0 as $n \to \infty$ for any $z \in \Omega$.
- ○ f_n converges uniformly to 0 on Ω as $n \to \infty$.
- ○ $\sum_{n=1}^{\infty} f_n(z)$ converges for any $z \in \Omega$.
- ○ $\sum_{n=1}^{\infty} f_n$ converges uniformly on Ω.

7. Given that f is holomorphic on $\mathbb{C} \backslash \{\alpha_1, \alpha_2, \alpha_3\}$ satisfying: **/2**

$$\text{Res}(f, \alpha_1) = 1 \qquad \text{Res}(f, \alpha_2) = 2 \qquad \text{Res}(f, \alpha_3) = 3$$

Suppose γ is simple closed counter-clockwise curve in $\mathbb{C}$ which *does not pass through* the points $\alpha_1, \alpha_2, \alpha_3$. List **ALL** possible value(s) of $\frac{1}{2\pi i} \oint_\gamma f(z)\, dz$:

8. Which of the following is/are correct? Put ✓ in **ALL** correct answer(s): **/2**

- ○ $1 + 2 + 3 + 4 + \cdots = -\frac{1}{12}$
- ○ $1 + 2 + 3 + 4 + \cdots$ diverges.
- ○ $\hat{\zeta}(-1) = -\frac{1}{12}$
- ○ $\hat{\zeta}(-1)$ is undefined.

9. Given a holomorphic function f on $B_R(z_0)$, it was proved in a homework problem that its Taylor's series about z_0 converges **uniformly** on a smaller ball $B_r(z_0)$ where $0 < r < R$. The key idea of the proof is to bound the remainder term: **/3**

$$R_N(z) := \frac{1}{2\pi i} \oint_{\gamma} \frac{f(\xi)}{\xi - z} \left(\frac{z - z_0}{\xi - z_0} \right)^N d\xi$$

where $\gamma := \partial B_{r+\varepsilon}(z_0)$ where $r < r + \varepsilon < R$.

Explain briefly why the same proof would not work if γ where chosen to be $\partial B_r(z_0)$.

10. Consider a function $f = u + iv$ which is holomorphic on a simply-connected domain Ω. If we *further assume* that f is C^1 on Ω, then Cauchy-Goursat's Theorem can be proved easily using Green's Theorem as follows: Let γ be a simply closed curve enclosing a region R, then: **/3**

$$\begin{aligned} \oint_{\gamma} f(z)\, dz &= \oint_{\gamma} (u + iv)\,(dx + idy) = \oint_{\gamma} (u\, dx - v\, dy) + i(v\, dx + u\, dy) \\ &= \iint_R \left(\frac{\partial(-v)}{\partial x} - \frac{\partial u}{\partial y} \right) dx\, dy + i \iint_R \left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y} \right) dx\, dy = 0. \end{aligned}$$

The last step uses the Cauchy-Riemann equations.

Green's Theorem requires f to be C^1, but any holomorphic function must be infinitely many times differentiable, so in particular, it must be C^1. *Therefore, the above proof of Cauchy-Goursat's Theorem using Green's Theorem is fully legitimate*. Do you agree with this claim? Explain briefly why or why not.

End of Part A

Part B - Long Questions (60 points)

Instructions:

- *Write your solutions in the yellow book provided. Clearly indicate the problem and part numbers. Open a new page for each problem.*
- *Problems are not necessarily arranged by the level of difficulty.*
- *You can directly quote any results proved or stated in the instructor's lecture notes (including results from exercises), tutorial notes, and homework problems.*

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1. Let a be a fixed real number such that $2a$ is not an integer. By considering the function **/12**

$$f(z) = \frac{\pi \cot \pi z}{(a+z)^2},$$

show that $\displaystyle\sum_{n=-\infty}^{\infty} \frac{1}{(a+n)^2} = \frac{\pi^2}{\sin^2 \pi a}$.

2. Let $f : \mathbb{C} \to \mathbb{C}$ be a continuous function. Define a complex-valued function F by the following integral over $t \in [0,1] \subset \mathbb{R}$:

$$F(z) = \int_0^1 t^{z-1} f(t)\, dt.$$

   (a) Show that $F(z)$ is holomorphic on the domain $\Omega := \{z \in \mathbb{C} : \mathrm{Re}(z) > 0\}$. Give full detail of your work, including why F is continuous on Ω. **/20**

   (b) Suppose further that f is holomorphic on $B_R(0)$ for some $R > 1$. Find the analytic continuation $\hat{F}$ of F on $\mathcal{O} := \mathbb{C}\backslash\{0, -1, -2, -3, \cdots\}$. **/12**
   Again, give full detail of your work, including why $\hat{F}$ is holomorphic on $\mathcal{O}$.

3. Let $f : \mathbb{C} \to \mathbb{C}$ be an entire function, whose Taylor's series about 0 is given by:

$$f(z) = a_0 + a_1 z + a_2 z^2 + \cdots = \sum_{n=0}^{\infty} a_n z^n.$$

   (a) Show that for any integer $n \geq 1$ and real number $r > 0$, we have: **/5**

$$\int_0^{2\pi} f(re^{i\theta}) \sin n\theta \, d\theta = i\pi a_n r^n$$

   (b) Two real numbers α and β are said to have the same sign if they are both positive, or both negative, or both zero. Suppose further that for each $z \in \mathbb{C}$, $\mathrm{Im}(f(z))$ and $\mathrm{Im}(z)$ always have the same sign.

      i. Show that $a_n \in \mathbb{R}$ for any $n \geq 0$. **/3**

      ii. Using results obtained above, show that **/8**

$$f(z) = a_0 + a_1 z$$

      for any $z \in \mathbb{C}$.
      [Hint and remark: You can use, without proofs, the fact that $n \sin\theta + \sin n\theta$, $n \sin\theta - \sin n\theta$, and $\sin\theta$ all have the same sign for any integer $n \geq 2$ and any $\theta \in [0, 2\pi]$.]

End of Exam