Chapter 5

De Rham Cohomology

> "Hydrodynamics procreated complex analysis, partial differential equations, Lie groups and algebra theory, cohomology theory and scientific computing."
>
> *Vladimir Arnold*

In Chapter 3, we discussed closed and exact forms. As a reminder, a smooth k-form ω on a smooth manifold M is *closed* if $d\omega = 0$ on M, and is *exact* if $\omega = d\eta$ for some smooth $(k-1)$-form η defined on the whole M.

By the fact that $d^2 = 0$, an exact form must be closed. It is then natural to ask whether every closed form is exact. The answer is no in general. Here is a counter-example. Let $M = \mathbb{R}^2 \backslash \{(0,0)\}$, and define

$$\omega := -\frac{y}{x^2+y^2}\,dx + \frac{x}{x^2+y^2}\,dy.$$

It can be computed easily that $d\omega = 0$ on M, and so ω is closed.

However, we can show that ω is *not* exact. Consider the unit circle C parametrized by $(x,y) = (\cos t, \sin t)$ where $0 < t < 2\pi$, and also the induced 1-form $\iota^*\omega$ (where $\iota : C \to M$). By direct computation, we get:

$$\oint_C \iota^*\omega = \int_0^{2\pi} -\frac{\sin t}{\cos^2 t + \sin^2 t}\,d(\cos t) + \frac{\cos t}{\cos^2 t + \sin^2 t}\,d(\sin t) = 2\pi.$$

If ω *were* exact, then $\omega = df$ for some smooth function $f : M \to \mathbb{R}$. Then, we would have:

$$\oint_C \iota^*\omega = \oint_C \iota^*(df) = \oint_C d(\iota^* f) = \int_0^{2\pi} \frac{d(\iota^* f)}{dt}\,dt.$$

Since $t = 0$ and $t = 2\pi$ represent the same point on C, by Fundamental Theorem of Calculus, we finally get:

$$\oint_C \iota^*\omega = 0$$

which is a contradiction! Therefore, ω is *not* exact on $\mathbb{R}^2 \backslash \{(0,0)\}$.

Heuristically, de Rham cohomology studies "how many" smooth k-forms defined on a given manifold M are closed but not exact. We should refine the meaning of "how many". Certainly, if η is any $(k-1)$-form on M, then $\omega + d\eta$ is also closed but not exact. Therefore, when we "count" how many smooth k-forms on M which are closed but not exact, it is fair to group ω and $\omega + d\eta$'s together, and counted them as one. In formal mathematical language, equivalence classes are used as we will discuss in detail. It turns out that the "number" of closed, not exact k-forms on a given M is a related to the *topology* of M!

In this chapter, we will learn the basics of de Rham cohomology, which is a beautiful topic to end the course MATH 4033.

5.1. De Rham Cohomology

Let M be a smooth manifold (with or without boundary). Recall that the exterior derivative d is a linear map that takes a k-form to a $(k+1)$-form, i.e. $d : \wedge^k T^*M \to \wedge^{k+1} T^*M$. We can then talk about the kernel and image of these maps. We define:

$$\begin{aligned}
\ker\left(d : \wedge^k T^*M \to \wedge^{k+1} T^*M\right) &= \{\omega \in \wedge^k T^*M : d\omega = 0\} \\
&= \{\text{closed } k\text{-forms on } M\} \\
\operatorname{Im}\left(d : \wedge^{k-1} T^*M \to \wedge^k T^*M\right) &= \{\omega \in \wedge^k T^*M : \omega = d\eta \text{ for some } \eta \in \wedge^{k-1} T^*M\} \\
&= \{\text{exact } k\text{-forms on } M\}
\end{aligned}$$

In many occasions, we may simply denote the above kernel and image by $\ker(d)$ and $\operatorname{Im}(d)$ whenever the value of k is clear from the context.

By $d^2 = 0$, it is easy to see that:

$$\operatorname{Im}\left(d : \wedge^{k-1} T^*M \to \wedge^k T^*M\right) \subset \ker\left(d : \wedge^k T^*M \to \wedge^{k+1} T^*M\right).$$

If all closed k-forms on a certain manifold are exact, then we have $\operatorname{Im}(d) = \ker(d)$. How "many" closed k-forms are exact is then measured by how $\operatorname{Im}(d)$ is "smaller" than $\ker(d)$, which is precisely measured by the size of the quotient vector space $\ker(d)/\operatorname{Im}(d)$. We call this quotient the de Rham cohomology group[1].

Definition 5.1 (de Rham Cohomology Group). Let M be a smooth manifold. For any positive integer k, we define the *k-th de Rham cohomology group of M* to be the quotient vector space:

$$H^k_{\mathrm{dR}}(M) := \frac{\ker\left(d : \wedge^k T^*M \to \wedge^{k+1} T^*M\right)}{\operatorname{Im}\left(d : \wedge^{k-1} T^*M \to \wedge^k T^*M\right)}.$$

Remark 5.2. When $k = 0$, then $\wedge^k T^*M = \wedge^0 T^*M = C^\infty(M, \mathbb{R})$ and $\wedge^{k-1} T^*M$ is not defined. Instead, we *define*

$$H^0_{\mathrm{dR}}(M) := \ker\left(d : C^\infty(M, \mathbb{R}) \to \wedge^1 T^*M\right) = \{f \in C^\infty(M, \mathbb{R}) : df = 0\},$$

which is the vector space of all locally constant functions on M. If M has N connected components, then a locally constant function f is determined by its value on each of the components. The space of functions $\{f : df = 0\}$ is in one-to-one correspondence an N-tuple $(k_1, \ldots, k_N) \in \mathbb{R}^N$, where k_i is the value of f on the i-th component of M. Therefore, $H^0_{\mathrm{dR}}(M) \simeq \mathbb{R}^N$ where N is the number of connected components of M. □

[1] A vector space is also a group whose addition is the vector addition. Although it is more appropriate or precise to call the quotient the "de Rham cohomology space", we will follow the tradition to call it a group.

5.1.1. Quotient Vector Spaces. Let's first review the basics about quotient vector spaces in Linear Algebra. Given a subspace W of a vector space V, we can define an equivalence relation $\sim$ by declaring that $v_1 \sim v_2$ if and only if $v_1 - v_2 \in W$. For example, if W is the x-axis and V is the xy-plane, then two vector v_1 and v_2 are equivalent under this relation if and only if they have the same j-component.

For each element $v \in V$ (the bigger space), one can define an equivalence class:

$$[v] := \{u \in V : u \sim v\} = \{u \in V : u - v \in W\}$$

which is the set of all vectors in V that are equivalent to v. For example, if W is the x-axis and V is $\mathbb{R}^2$, then the class $[(2,3)]$ is given by:

$$[(2,3)] = \{(x,3) : x \in \mathbb{R}\}$$

which is the horizontal line $\{y = 3\}$. Similarly, one can figure out $[(1,3)] = [(2,3)] = [(3,3)] = \ldots$ as well, but $[(2,3)] \neq [(2,2)]$, and the latter is the line $\{y = 2\}$.

The quotient space V/W is defined to be the set of all equivalence classes, i.e.

$$V/W := \{[v] : v \in V\}.$$

For example, if V is $\mathbb{R}^2$ and W is the x-axis, then V/W is the set of all horizontal lines in $\mathbb{R}^2$. For finite dimensional vector spaces, one can show (see Exercise 5.1) that

$$\dim(V/W) = \dim V - \dim W,$$

and so the "size" (precisely, the dimension) of the quotient V/W measures how small W is when compared to V. In fact, if the bases of V and W are suitably chosen, we can describe the basis of V/W in a precise way (see Exercise 5.1).

Exercise 5.1. Let W be a subspace of a finite dimensional vector space V. Suppose $\{w_1, \ldots, w_k\}$ is a basis for W, and $\{w_1, \ldots, w_k, v_1, \ldots, v_l\}$ is a basis for V (Remark: given any basis $\{w_1, \ldots, w_k\}$ for the subspace W, one can always complete it to form a basis for V).

(a) Show that given any vector $\sum_{i=1}^{k} \alpha_i w_i + \sum_{j=1}^{l} \beta_j v_j \in V$, the equivalence class represented by this vector is given by:

$$\left[\sum_{i=1}^{k} \alpha_i w_i + \sum_{j=1}^{l} \beta_j v_j\right] = \left\{\sum_{i=1}^{k} \gamma_i w_i + \sum_{j=1}^{l} \beta_j v_j : \gamma_i \in \mathbb{R}\right\} = \left[\sum_{j=1}^{l} \beta_j v_j\right].$$

(b) Hence, show that $\{[v_1], \ldots, [v_l]\}$ is a basis for V/W, and so

$$\dim V/W = l = \dim V - \dim W.$$

Exercise 5.2. Given a subspace W of a vector space V, and define an equivalence relation $\sim$ by declaring that $v_1 \sim v_2$ if and only if $v_1 - v_2 \in W$. Show that the following are equivalent:

(1) $u \in [v]$

(2) $u - v \in W$

(3) $[u] = [v]$

5.1.2. Cohomology Classes and Betti numbers. Recall that the k-th de Rham cohomology group $H^k_{\mathrm{dR}}(M)$, where $k \geq 1$, of a smooth manifold M is defined to be the quotient vector space:

$$H^k_{\mathrm{dR}}(M) := \frac{\ker\left(d : \wedge^k T^*M \to \wedge^{k+1} T^*M\right)}{\mathrm{Im}\,\left(d : \wedge^{k-1} T^*M \to \wedge^k T^*M\right)}.$$

Given a closed k-form ω, we then define its equivalence class to be:

$$\begin{aligned}[\omega] &:= \{\omega' : \omega' - \omega \text{ is exact}\} \\ &= \{\omega' : \omega' = \omega + d\eta \text{ for some } \eta \in \wedge^{k-1}T^*M\} \\ &= \{\omega + d\eta : \eta \in \wedge^{k-1}T^*M\}.\end{aligned}$$

An equivalence class $[\omega]$ is called the *de Rham cohomology class* represented by (or containing) ω, and ω is said to be a representative of this de Rham cohomology class.

By Exercise 5.1, its dimension is given by

$$\begin{aligned}&\dim H^k_{\mathrm{dR}}(M) \\ &= \dim\ker\left(d : \wedge^k T^*M \to \wedge^{k+1}T^*M\right) - \dim\mathrm{Im}\left(d : \wedge^{k-1}T^*M \to \wedge^k T^*M\right)\end{aligned}$$

provided that both kernel and image are finite-dimensional.

Therefore, the dimension of $H^k_{\mathrm{dR}}(M)$ is a measure of "how many" closed k-forms on M are not exact. Due to the importance of this dimension, we have a special name for it:

Definition 5.3 (Betti Numbers)**.** Let M be a smooth manifold. The *k-th Betti number* of M is defined to be:

$$b_k(M) := \dim H^k_{\mathrm{dR}}(M).$$

In particular, $b_0(M) = \dim H^0_{\mathrm{dR}}(M)$ is the number of connected components of M.

In case when $M = \mathbb{R}^2\backslash\{(0,0)\}$, we discussed that there is a closed 1-form

$$\omega = \frac{-y\,dx + x\,dy}{x^2 + y^2}$$

defined on M which is not exact. Therefore, $\omega \in \ker\left(d : \wedge^1 T^*M \to \wedge^2 T^*M\right)$ yet $\omega \notin \mathrm{Im}\left(d : \wedge^0 T^*M \to \wedge^1 T^*M\right)$, and so in $H^1_{\mathrm{dR}}(M)$ we have $[\omega] \neq [0]$. From here we can conclude that $H^1_{\mathrm{dR}}(M) \neq \{[0]\}$ and $b_1(M) \geq 1$. We will later show that in fact $b_1(M) = 1$ using some tools in later sections.

Exercise 5.3. If $k > \dim M$, what can you say about $b_k(M)$?

5.1.3. Poincaré Lemma.

A *star-shaped* open set U in $\mathbb{R}^n$ is a region containing a point $\mathsf{p} \in U$ (call it a base point) such that any line segment connecting a point $\mathsf{x} \in U$ and the base point p must be contained inside U. Examples of star-shaped open sets include convex open sets such an open ball $\{\mathsf{x} \in \mathbb{R}^n : |x| < 1\}$, and all of $\mathbb{R}^n$. The following Poincaré Lemma asserts that $H^1_{\mathrm{dR}}(U) = \{[0]\}$.

Theorem 5.4 (Poincaré Lemma for H^1_{dR})**.** *For any star-shaped open set U in $\mathbb{R}^n$, we have $H^1_{\mathrm{dR}}(U) = \{[0]\}$. In other words, any closed 1-form defined on a star-shaped open set is exact on that open set.*

Proof. Given a closed 1-form ω defined on U, given by $\omega = \sum_i \omega_i\,dx^i$, we need to find a smooth function $f : U \to \mathbb{R}$ such that $\omega = df$. In other words, we need $\dfrac{\partial f}{\partial x_i} = \omega_i$ for any i.

Let p be the base point of U, then given any $\mathsf{x} \in U$, we define:

$$f(\mathsf{x}) := \int_{L_{\mathsf{x}}} \omega$$

where L_{x} is the line segment joining p and x, which can be parametrized by:

$$\mathsf{r}(t) = (1-t)\mathsf{p} + t\mathsf{x}, \quad t \in [0,1].$$

Write $\mathsf{p} = (p_1, \ldots, p_n)$, $\mathsf{x} = (x_1, \ldots, x_n)$, then $f(x)$ can be expressed in terms of t by:

$$f(\mathsf{x}) = \int_0^1 \sum_{i=1}^n \omega_i(\mathsf{r}(t)) \cdot (x_i - p_i)\, dt.$$

Using the chain rule, we can directly verify that:

$$\begin{aligned}
\frac{\partial f}{\partial x_j}(\mathsf{x}) &= \frac{\partial}{\partial x_j} \int_0^1 \sum_{i=1}^n \omega_i(\mathsf{r}(t)) \cdot (x_i - p_i)\, dt \\
&= \sum_{i=1}^n \int_0^1 \left(\frac{\partial}{\partial x_j} \omega_i(\mathsf{r}(t)) \cdot (x_i - p_i) + \omega_i(\mathsf{r}(t)) \cdot \frac{\partial}{\partial x_j}(x_i - p_i) \right) dt \\
&= \sum_{i=1}^n \int_0^1 \left(\sum_{k=1}^n \frac{\partial \omega_i}{\partial x_k} \frac{\partial \overbrace{((1-t)p_k + tx_k)}^{x_k \circ \mathsf{r}(t)}}{\partial x_j} \cdot (x_i - p_i) + \omega_i(\mathsf{r}(t)) \cdot \delta_{ij} \right) dt \\
&= \sum_{i=1}^n \int_0^1 \left(\sum_{k=1}^n t \frac{\partial \omega_i}{\partial x_k} \delta_{jk} \cdot (x_i - p_i) + \omega_j(\mathsf{r}(t)) \right) dt \\
&= \sum_{i=1}^n \int_0^1 \left(t \frac{\partial \omega_i}{\partial x_j} \cdot (x_i - p_i) + \omega_j(\mathsf{r}(t)) \right) dt
\end{aligned}$$

Since ω is closed, we have:

$$0 = d\omega = \sum_{i<j}^n \left(\frac{\partial \omega_i}{\partial x_j} - \frac{\partial \omega_j}{\partial x_i} \right) dx^j \wedge dx^i$$

and hence $\dfrac{\partial \omega_i}{\partial x_j} = \dfrac{\partial \omega_j}{\partial x_i}$ for any i, j. Using this to proceed our calculation:

$$\begin{aligned}
\frac{\partial f}{\partial x_j}(\mathsf{x}) &= \int_0^1 \left(t \frac{\partial \omega_j}{\partial x_i} \cdot (x_i - p_i) + \omega_j(\mathsf{r}(t)) \right) dt \\
&= \int_0^1 \frac{d}{dt} \left(t\omega_j(\mathsf{r}(t)) \right) dt \\
&= \left[t\omega_j(\mathsf{r}(t)) \right]_{t=0}^{t=1} = \omega_j(\mathsf{r}(1)) = \omega_j(\mathsf{x}).
\end{aligned}$$

In the second equality above, we have used the chain rule backward:

$$\frac{d}{dt} \left(t\omega_j(\mathsf{r}(t)) \right) = t \frac{\partial \omega_j}{\partial x_i} \cdot (x_i - p_i) + \omega_j(\mathsf{r}(t)).$$

From this, we conclude that $\omega = df$ on U, and hence $[\omega] = [0]$ in $H^1_{\mathrm{dR}}(U)$. Since ω is an arbitrary closed 1-form on U, we have $H^1_{\mathrm{dR}}(U) = \{[0]\}$. □

Remark 5.5. Poincaré Lemma also holds for H^k_{dR}, meaning that if U is a star-shaped open set in $\mathbb{R}^n$, then $H^k_{\mathrm{dR}}(U) = \{[0]\}$ for any $k \geq 1$. However, the proof involves the use of Lie derivatives and a formula by Cartan, both of which are beyond the scope of this course. Note also that $H^0_{\mathrm{dR}}(U) \simeq \mathbb{R}$ since a star-shaped open set must be connected. □

Remark 5.6. We have discussed that the 1-form

$$\omega = \frac{-y\,dx + x\,dy}{x^2 + y^2}$$

is closed but not exact. To be precise, it is not exact *on* $\mathbb{R}^2 \backslash \{(0,0)\}$. However, if we regard the domain to be the first quadrant $U := \{(x,y) : x > 0 \text{ and } y > 0\}$, which is a star-shaped open set in $\mathbb{R}^2$, then by Poinaré Lemma (Theorem 5.4), ω is indeed an exact 1-form on U. In fact, it is not difficult to verify that

$$\omega = d\left(\tan^{-1}\frac{y}{x}\right) \quad \text{on } U.$$

Note that the scalar function $\tan^{-1}\frac{y}{x}$ is smoothly defined on U. Whether a form is exact or not depends on the choice of its domain! □

5.1.4. Diffeomorphic Invariance. By Proposition 3.50, we learned that the exterior derivative d commutes with the pull-back of a smooth map between two manifolds. An important consequence is that the de Rham cohomology group is invariant under diffeomorphism.

Let $\Phi : M \to N$ be any smooth map between two smooth manifolds. The pull-back map $\Phi^* : \wedge^k T^*N \to \wedge^k T^*M$ induces a well-defined pull-back map (which is also denoted by Φ^*) from $H^k_{\mathrm{dR}}(N)$ to $H^k_{\mathrm{dR}}(M)$. Precisely, given any closed k-form ω on N, we define:

$$\Phi^*[\omega] := [\Phi^*\omega].$$

$\Phi^*\omega$ is a k-form on M. It is closed since $d(\Phi^*\omega) = \Phi^*(d\omega) = \Phi^*(0) = 0$. To show it is well-defined, we take another k-form ω' on N such that $[\omega'] = [\omega]$ in $H^k_{\mathrm{dR}}(N)$. Then, there exists a $(k-1)$-form η on N such that:

$$\omega' - \omega = d\eta \quad \text{on } N.$$

Using again $d \circ \Phi^* = \Phi^* \circ d$, we get:

$$\Phi^*\omega' - \Phi^*\omega = \Phi^*(d\eta) = d(\Phi^*\eta) \quad \text{on } M$$

We conclude $\Phi^*\omega' - \Phi^*\omega$ is exact and so

$$[\Phi^*\omega'] = [\Phi^*\omega] \text{ in } H^k_{\mathrm{dR}}(M).$$

This shows $\Phi^* : H^k_{\mathrm{dR}}(N) \to H^k_{\mathrm{dR}}(M)$ is a well-defined map.

Theorem 5.7 (Diffeomorphism Invariance of H^k_{dR})**.** *If two smooth manifolds M and N are diffeomorphic, then $H^k_{\mathrm{dR}}(M)$ and $H^k_{\mathrm{dR}}(N)$ are isomorphic for any $k \geq 0$.*

Proof. Let $\Phi : M \to N$ be a diffeomorphism, then $\Phi^{-1} : N \to M$ exists and we have $\Phi \circ \Phi^{-1} = \mathrm{id}_N$ and $\Phi^{-1} \circ \Phi = \mathrm{id}_M$. By the chain rule for tensors (Theorem 3.47), we have:

$$(\Phi^{-1})^* \circ \Phi^* = \mathrm{id}_{\wedge^k T^*N} \quad \text{and} \quad \Phi^* \circ (\Phi^{-1})^* = \mathrm{id}_{\wedge^k T^*M}.$$

Given any closed k-form ω on M, then in $H^k_{\mathrm{dR}}(M)$ we have:

$$\Phi^* \circ (\Phi^{-1})^*[\omega] = \Phi^*[(\Phi^{-1})^*\omega] = [\Phi^* \circ (\Phi^{-1})^*\omega] = [\omega].$$

In other words, $\Phi^* \circ (\Phi^{-1})^*$ is also the identity map of $H^k_{\mathrm{dR}}(M)$. Similarly, one can also show $(\Phi^{-1})^* \circ \Phi^*$ is the identity map of $H^k_{\mathrm{dR}}(N)$. Therefore, $H^k_{\mathrm{dR}}(M)$ and $H^k_{\mathrm{dR}}(N)$ are isomorphic (as vector spaces). □

Corollary 5.8. *Given any smooth manifold M which is diffeomorphic to a star-shaped open set in $\mathbb{R}^n$, we have $H^1_{\mathrm{dR}}(M) \simeq \{[0]\}$, or in other words, every closed 1-form ω on such a manifold M is exact.*

Proof. Combine the results of the Poincaré Lemma (Theorem 5.4) and the diffeomorphism invariance of H^1_{dR} (Theorem 5.7). □

Consequently, a large class of open sets in $\mathbb{R}^n$ has trivial H^1_{dR} as long as it is diffeomorphic to a star-shaped manifold. For open sets in $\mathbb{R}^2$, there is a celebrated result called Riemann Mapping Theorem, which says any (non-empty) simply-connected open bounded subset U in $\mathbb{R}^2$ is diffeomorphic to the unit open ball in $\mathbb{R}^2$. In fact, the diffeomorphism can be chosen so that angles are preserved, but we don't need this when dealing with de Rham cohomology.

Under the assumption of Riemann Mapping Theorem (whose proof can be found in advanced Complex Analysis textbooks), we can establish that $H^1_{\mathrm{dR}}(U) = \{[0]\}$ for any (non-empty) simply-connected subset U in $\mathbb{R}^2$. Consequently, any closed 1-form on such a domain U is exact on U. Using the language in Multivariable Calculus (or Physics), this means any curl-zero vector field defined on a (non-empty) simply-connected domain U in $\mathbb{R}^2$ must be conservative on U. You might have learned this fact without proof in MATH 2023.

5.2. Deformation Retracts

In the previous section, we learned that two diffeomorphic manifolds have isomorphic de Rham cohomology groups. In short, we say de Rham cohomology is a diffeomorphic invariance. In this section, we will discuss another type of invariance: *deformation retracts*.

Let M be a smooth manifold (with or without boundary), and Σ is a submanifold of M. Note that Σ can have lower dimension than M. Roughly speaking, we say Σ is a *deformation retract* of M if one can continuously contract M onto Σ. Let's make it more precise:

Definition 5.9 (Deformation Retract)**.** Let M be a smooth manifold, and Σ is a submanifold of M. If there exists a C^1 family of smooth maps $\{\Psi_t : M \to M\}_{t\in[0,1]}$ satisfying all three conditions below:

- $\Psi_0(x) = x$ for any $x \in M$, i.e. $\Psi_0 = \mathrm{id}_M$;
- $\Psi_1(x) \in \Sigma$ for any $x \in M$, i.e. $\Psi_1 : M \to \Sigma$;
- $\Psi_t(p) = p$ for any $p \in \Sigma$, $t \in [0,1]$, i.e. $\Psi_t|_\Sigma = \mathrm{id}_\Sigma$ for any $t \in [0,1]$,

then we say Σ is a deformation retract of M. Equivalently, we can also say M deformation retracts onto Σ.

One good way to think of a deformation retract is to regard t as the time, and Ψ_t is a "movie" that demonstates how M collapses onto Σ. The condition $\Psi_0 = \mathrm{id}_M$ says initially (at $t = 0$), the "movie" starts with the image M. At the final scene (at $t = 1$), the condition $\Psi_1 : M \to \Sigma$ says that the image eventually becomes Σ . The last condition $\Psi_t(p) = p$ for any $p \in \Sigma$ means the points on Σ do not move throughout the movie. Before we talk about the relation between cohomology and deformation retract, let's first look at some examples:

Example 5.10. The unit circle $\mathbb{S}^1$ defined by $\{(x,y) : x^2 + y^2 = 1\}$ is a deformation retract of the annulus $\{(x,y) : \frac{1}{4} < x^2 + y^2 < 4\}$. To describe such a retract, it's best to use polar coordinates:

$$\Psi_t(re^{i\theta}) = (r + t(1-r))\, e^{i\theta}$$

For each $t \in [0,1]$, the map Ψ_t has image inside the annulus since $r + t(1-r) \in (\frac{1}{2}, 2)$ whenever $r \in (\frac{1}{2}, 2)$ and $t \in [0,1]$. One can easily check that $\Psi_0(re^{i\theta}) = re^{i\theta}$, $\Psi_1(re^{i\theta}) = e^{i\theta}$ and $\Psi_t(e^{i\theta}) = e^{i\theta}$ for any (r,θ) and $t \in [0,1]$. Hence Ψ_t fulfills all three conditions stated in Definition 5.9. □

Example 5.11. Intuitively, we can see the letters E, F, H, K, L, M and N all deformation retract onto the letter I. Also, the letter Q deformation retracts onto the letter O. The explicit Ψ_t for each deformation retract is not easy to write down. □

Example 5.12. A two-dimensional torus with a point removed can deformation retract onto two circles joined at one point. Try to visualize it! □

Exercise 5.4. Show that the unit circle $x^2 + y^2 = 1$ in $\mathbb{R}^2$ is a deformation retract of $\mathbb{R}^2 \backslash \{(0,0)\}$.

Exercise 5.5. Show that any star-shaped open set U in $\mathbb{R}^n$ deformation retracts onto its base point.

Exercise 5.6. Let M be a smooth manifold, and Σ_0 be the zero section of the tangent bundle, i.e. Σ_0 consists of all pairs $(p, 0_p)$ in TM where $p \in M$ and 0_p is the zero vector in T_pM. Show that the zero section Σ_0 is a deformation retract of the tangent bundle TM.

Exercise 5.7. Define a relation $\sim$ of manifolds by declaring that $M_1 \sim M_2$ if and only if M_1 is a deformation retract of M_2. Is $\sim$ an equivalence relation?

We next show an important result in de Rham theory, which asserts that deformation retracts preserve the first de Rham cohomology group.

Theorem 5.13 (Invariance under Deformation Retracts). *Let M be a smooth manifold, and Σ be a submanifold of M. If Σ is a deformation retract of M, then $H^1_{\mathrm{dR}}(M)$ and $H^1_{\mathrm{dR}}(\Sigma)$ are isomorphic.*

Proof. Let $\iota : \Sigma \to M$ be the inclusion map, and $\{\Psi_t : M \to M\}_{t\in[0,1]}$ be the family of maps satisfying all conditions stated in Definition 5.9. Then, the pull-back map $\iota^* : \wedge^1 T^*M \to \wedge^1 T^*\Sigma$ induces a map $\iota^* : H^1_{\mathrm{dR}}(M) \to H^1_{\mathrm{dR}}(\Sigma)$. Also, the map $\Psi_1 : M \to \Sigma$ induces a pull-back map $\Psi_1^* : H^1_{\mathrm{dR}}(\Sigma) \to H^1_{\mathrm{dR}}(M)$. The key idea of the proof is to show that Ψ_1^* and ι^* are inverses of each other as maps between $H^1_{\mathrm{dR}}(M)$ and $H^1_{\mathrm{dR}}(\Sigma)$.

Let ω be an arbitrary closed 1-form defined on M. Similar to the proof of Poincaré Lemma (Theorem 5.4), we consider the scalar function $f : M \to \mathbb{R}$ defined by:

$$f(x) = \int_{\Psi_t(x)} \omega$$

Here, $\Psi_t(x)$ is regarded as a curve with parameter t joining $\Psi_0(x) = x$ and $\Psi_1(x) \in \Sigma$. We will show the following result:

$$\Psi_1^* \iota^* \omega - \omega = df \tag{5.1}$$

which will imply $[\omega] = \Psi_1^* \iota^* [\omega]$, or in other words, $\Psi_t^* \circ \iota^* = \mathrm{id}$ on $H^1_{\mathrm{dR}}(M)$.

To prove (5.1), we use local coordinates $(u_1, \ldots, u_n)$, and express ω in terms of local coordinates $\omega = \sum_i \omega_i du^i$. For simplicity, let's assume that such a local coordinate chart can cover the whole curve $\Psi_t(x)$ for $t \in [0,1]$. We will fix this issue later. For each $t \in [0,1]$, we write $\Psi_t^i(x)$ to be the u_i-coordinate of $\Psi_t(x)$, i.e. $\Psi_t^i = u_i \circ \Psi_t$. Then, one can calculate df using local coordinates. The calculation is similar to the one we did in the proof of Poincaré Lemma (Theorem 5.4):

$$\begin{aligned}
f(x) &= \int_{\Psi_t(x)} \omega = \int_0^1 \sum_i \omega_i(\Psi_t(x)) \frac{\partial \Psi_t^i}{\partial t}\, dt \\
(df)(x) &= \sum_j \frac{\partial f}{\partial u_j} du^j = \sum_j \left\{ \int_0^1 \frac{\partial}{\partial u_j} \left(\sum_i \omega_i(\Psi_t(x)) \frac{\partial \Psi_t^i}{\partial t} \right) dt \right\} du^j \\
&= \sum_j \left\{ \int_0^1 \left[\sum_{i,k} \left.\frac{\partial \omega_i}{\partial u_k}\right|_{\Psi_t(x)} \frac{\partial \Psi_t^k}{\partial u_j} \frac{\partial \Psi_t^i}{\partial t} + \sum_i \omega_i(\Psi_t(x)) \frac{\partial}{\partial t} \left(\frac{\partial \Psi_t^i}{\partial u_j} \right) \right] dt \right\} du^j
\end{aligned}$$

Next, recall that ω is a closed 1-form, so we have $\dfrac{\partial \omega_i}{\partial u_k} = \dfrac{\partial \omega_k}{\partial u_i}$ for any i, k. Using this on the first term, and by switching indices of the second term in the integrand, we get:

$$\begin{aligned}(df)(x) &= \sum_j \left\{ \int_0^1 \left[\sum_{i,k} \frac{\partial \omega_k}{\partial u_i}\bigg|_{\Psi_t(x)} \frac{\partial \Psi_t^k}{\partial u_j} \frac{\partial \Psi_t^i}{\partial t} + \sum_k \omega_k(\Psi_t(x)) \frac{\partial}{\partial t}\left(\frac{\partial \Psi_t^k}{\partial u_j}\right)\right] dt \right\} du^j \\ &= \sum_j \left\{ \int_0^1 \frac{\partial}{\partial t}\left(\sum_k \omega_k(\Psi_t(x)) \frac{\partial \Psi_t^k}{\partial u_j}\right) dt \right\} du^j = \sum_{j,k} \left[\omega_k(\Psi_t(x)) \frac{\partial \Psi_t^k}{\partial u_j}\right]_{t=0}^{t=1} du^j\end{aligned}$$

where the last equality follows from the (backward) chain rule.

Denote $\iota_t : \Psi_t(M) \to M$ the inclusion map at time t, then one can check that

$$\begin{aligned}\Psi_t^* \iota_t^* \omega(x) = (\iota_t \circ \Psi_t)^* \omega(x) &= (\iota_t \circ \Psi_t)^* \sum_k \omega_k du^k \\ &= \sum_k \omega_k(\iota_t \circ \Psi_t(x))\, d(u_k \circ \iota_t \circ \Psi_t(x)) \\ &= \sum_k \omega_k(\iota_t \circ \Psi_t(x))\, d\Psi_t^k \\ &= \sum_{j,k} \omega_k(\Psi_t(x))\, \frac{\partial \Psi_t^k}{\partial u_j}\, du^j.\end{aligned}$$

Therefore, we get:

$$df = \sum_{j,k} \left[\omega_k(\Psi_t(x)) \frac{\partial \Psi_t^k}{\partial u_j}\right]_{t=0}^{t=1} du^j = [\Psi_t^* \iota_t^* \omega]_{t=0}^{t=1} = \Psi_1^* \iota_1^* \omega - \Psi_0^* \iota_0^* \omega.$$

Since $\Psi_0 = \mathrm{id}_M$ and $\iota_0 = \mathrm{id}_M$, we have proved (5.1). In case $\Psi_t(x)$ cannot be covered by one single local coordinate chart, one can then modify the above proof a bit by covering the curve $\Psi_t(x)$ by finitely many local coordinate charts. It can be done because $\Psi_t(x)$ is compact. Suppose $0 = t_0 < t_1 < \ldots < t_N = 1$ is a partition of $[0, 1]$ such that for each α, the curve $\Psi_t(x)$ restricted to $t \in [t_{\alpha-1}, t_\alpha]$ can be covered by a single local coordinate chart, then we have:

$$f(x) = \sum_{\alpha=1}^{N} \int_{t_{\alpha-1}}^{t_\alpha} \sum_i \omega_i(\Psi_t(x)) \frac{\partial \Psi_t^i}{\partial t}\, dt.$$

Proceed as in the above proof, we can get:

$$df = \sum_{\alpha=1}^{N} \left(\Psi_{t_\alpha}^* \iota_{t_\alpha}^* \omega - \Psi_{t_\alpha - 1}^* \iota_{t_\alpha - 1}^* \omega \right) = \Psi_1^* \iota_1^* \omega - \Psi_0^* \iota_0^* \omega,$$

which completes the proof of (5.1) in the general case.

To complete the proof of the theorem, we consider an arbitrary 1-form η on Σ. We claim that

$$\iota^* \Psi_1^* \eta = \eta. \tag{5.2}$$

We prove by direct verification using local coordinates $(u_1, \ldots, u_n)$ on M such that:

$$(u_1, \ldots, u_k, 0, \ldots, 0) \in \Sigma.$$

Such a local coordinate system always exists near Σ by Immersion Theorem (Theorem 2.42). Locally, denote $\eta = \sum_{i=1}^{k} \eta_i \, du^i$, then

$$\begin{aligned}(\Psi_1^*\eta)(x) &= \sum_{i=1}^{k} \Psi_1^*(\eta_i(x)\, du^i) = \sum_{i=1}^{k} \eta_i(\Psi_1(x))\, d(u^i \circ \Psi_1) \\ &= \sum_{i=1}^{k}\sum_{j=1}^{k} \eta_i(\Psi_1(x)) \frac{\partial \Psi_1^i(x)}{\partial u_j}\, du^j.\end{aligned}$$

Since $\Psi_1(x) = x$ whenever $x \in \Sigma$, we have $\Psi_1^i(x) = u_i(x)$ where $u_i(x)$ is the i-th coordinate of x. Therefore, we get $\dfrac{\partial \Psi_1^i(x)}{\partial u_j} = \dfrac{\partial u_i}{\partial u_j} = \delta_{ij}$ and so:

$$(\Psi_1^*\eta)(x) = \sum_{i,j=1}^{k} \eta_i(x)\delta_{ij}\, du^j = \sum_{i=1}^{k} \eta_i(x)\, du^i = \eta(x)$$

for any $x \in \Sigma$. In other words, $\iota^*\Psi_1^*\eta = \eta$ on Σ. This proves (5.2).

Combining (5.1) and (5.2), we get $\iota^* \circ \Psi_1^* = \mathrm{id}$ on $H^1_{\mathrm{dR}}(\Sigma)$, and $\Psi_1^* \circ \iota^* = \mathrm{id}$ on $H^1_{\mathrm{dR}}(M)$. As a result, Ψ_1^* and ι^* are inverses of each other in H^1_{dR}. It completes the proof that $H^1_{\mathrm{dR}}(M)$ and $H^1_{\mathrm{dR}}(\Sigma)$ are isomorphic. □

Using Theorem 5.13, we see that $H^1_{\mathrm{dR}}(\mathbb{R}^2\backslash\{(0,0)\})$ and $H^1_{\mathrm{dR}}(\mathbb{S}^1)$ are isomorphic, and hence $b_1(\mathbb{R}^2\backslash\{(0,0)\}) = b_1(\mathbb{S}^1)$. At this moment, we still don't know the exact value of $b_1(\mathbb{S}^1)$, but we will figure it out in the next section.

Note that Theorem 5.13 holds for H^k_{dR} for any $k \geq 2$ as well, but the proof again uses some Lie derivatives and Cartan's formula, which are beyond the scope of this course.

Another nice consequence of Theorem 5.13 is the 2-dimensional case of the following celebrated theorem in topology:

Theorem 5.14 (Brouwer's Fixed-Point Theorem on $\mathbb{R}^2$)**.** *Let $B_1(0)$ be the closed ball with radius 1 centered at origin in $\mathbb{R}^2$. Suppose $\Phi : B_1(0) \to B_1(0)$ is a smooth map between $B_1(0)$. Then, there exists a point $\mathsf{x} \in B_1(0)$ such that $\Phi(\mathsf{x}) = \mathsf{x}$.*

Proof. We prove by contradiction. Suppose $\Phi(\mathsf{x}) \neq \mathsf{x}$ for any $\mathsf{x} \in B_1(0)$. Then, we let $\Psi_t(\mathsf{x})$ be a point in $B_1(0)$ defined in the following way:

(1) Consider the vector $\mathsf{x} - \Phi(\mathsf{x})$ which is non-zero.
(2) Consider the straight ray emanating from x in the direction of $\mathsf{x} - \Phi(\mathsf{x})$. This ray will intersect the unit circle $\mathbb{S}^1$ at a unique point p_x.
(3) We then define $\Psi_t(\mathsf{x}) := (1-t)\mathsf{x} + t\mathsf{p}_\mathsf{x}$

We leave it as an exercise for readers to write down the explicit formula for $\Psi_t(\mathsf{x})$, and show that it is smooth for each $t \in [0,1]$.

Clearly, we have $\Psi_0(\mathsf{x}) = \mathsf{x}$ for any $\mathsf{x} \in B_1(0)$; $\Psi_1(\mathsf{x}) = \mathsf{p}_\mathsf{x} \in \mathbb{S}^1$; and if $|\mathsf{x}| = 1$, then $\mathsf{p}_\mathsf{x} = \mathsf{x}$ and so $\Psi_t(\mathsf{x}) = \mathsf{x}$.

Therefore, it shows $\mathbb{S}^1$ is a deformation retract of $B_1(0)$, and by Theorem 5.13, their H^1_{dR}'s are isomorphic. However, we know $H^1_{\mathrm{dR}}(B_1(0)) \simeq \{[0]\}$, while $H^1_{\mathrm{dR}}(\mathbb{S}^1) \simeq H^1_{\mathrm{dR}}(\mathbb{R}^2\backslash\{(0,0)\}) \neq \{[0]\}$. It is a contradiction! It completes the proof that there is at least a point $\mathsf{x} \in B_1(0)$ such that $\Phi(\mathsf{x}) = \mathsf{x}$. □

Exercise 5.8. Write down an explicit expression of p_{x} in the above proof, and hence show that Ψ_t is smooth for each fixed t.

Exercise 5.9. Generalize the Brouwer's Fixed-Point Theorem in the following way: given a manifold Ω which is diffeomorphic to $B_1(0)$, and a smooth map $\Phi : \Omega \to \Omega$. Using Theorem 5.14, show that there exists a point $p \in \Omega$ such that $\Phi(p) = p$.

Exercise 5.10. What fact(s) are needed to be established in order to prove the Brouwer's Fixed-Point Theorem for general $\mathbb{R}^n$ using a similar way as in the proof of Theorem 5.14?

5.3. Mayer-Vietoris Theorem

In the previous section, we showed that if Σ is a deformation retract of M, then $H^1_{\text{dR}}(\Sigma)$ and $H^1_{\text{dR}}(M)$ are isomorphic. For instance, this shows $H^1_{\text{dR}}(\mathbb{S}^1)$ is isomorphic to $H^1_{\text{dR}}(\mathbb{R}^2\backslash\{(0,0)\})$. Although we have discussed that $H^2_{\text{dR}}(\mathbb{R}^2\backslash\{(0,0)\})$ is non-trivial, we still haven't figured out what this group is. In this section, we introduce a useful tool, called Mayer-Vietoris sequence, that we can use to compute the de Rham cohomology groups of $\mathbb{R}^2\backslash\{(0,0)\}$, as well as many other spaces.

5.3.1. Exact Sequences. Consider a sequence of homomorphism between abelian groups:

$$\cdots \xrightarrow{T_{k-1}} G_{k-1} \xrightarrow{T_k} G_k \xrightarrow{T_{k+1}} G_{k+1} \xrightarrow{G_{k+1}} \cdots$$

We say it is an *exact sequence* if the image of each homomorphism is equal to the kernel of the next one, i.e.

$$\operatorname{Im} T_{i-1} = \ker T_i \quad \text{for each } i.$$

One can also talk about exact-ness for a finite sequence, say:

$$G_0 \xrightarrow{T_1} G_1 \xrightarrow{T_2} G_2 \xrightarrow{T_3} \cdots \xrightarrow{T_{n-1}} G_{n-1} \xrightarrow{T_n} G_n$$

However, such a T_1 would not have a previous map, and such an T_n would not have the next map. Therefore, whenever we talk about the exact-ness of a finite sequence of maps, we will add two trivial maps at both ends, i.e.

$$0 \xrightarrow{0} G_0 \xrightarrow{T_1} G_1 \xrightarrow{T_2} G_2 \xrightarrow{T_3} \cdots G_{n-1} \xrightarrow{T_n} G_n \xrightarrow{0} 0. \tag{5.3}$$

The first map $0 \xrightarrow{0} G_0$ is the homomorphism taking the zero in the trivial group to the zero in G_0. The last map $G_n \xrightarrow{0} 0$ is the linear map that takes every element in G_n to the zero in the trivial group. We say the finite sequence (5.3) an *exact sequence* if

$$\operatorname{Im}(0 \xrightarrow{0} G_0) = \ker T_1, \quad \operatorname{Im} T_n = \ker(G_n \xrightarrow{0} 0), \text{ and } \operatorname{Im} T_i = \ker T_{i+1} \quad \text{for any } i.$$

Note that $\operatorname{Im}(0 \xrightarrow{0} G_0) = \{0\}$ and $\ker(G_n \xrightarrow{0} 0) = G_n$, so if (5.3) is an exact sequence, it is necessary that

$$\ker T_1 = \{0\} \quad \text{and} \quad \operatorname{Im} T_n = G_n$$

or equivalently, T_1 is injective and T_n is surjective.

One classic example of a finite exact sequence is:

$$0 \to \mathbb{Z} \xrightarrow{\iota} \mathbb{C} \xrightarrow{f} \mathbb{C}\backslash\{0\} \to 0$$

where $\iota : \mathbb{Z} \to \mathbb{C}$ is the inclusion map taking $n \in \mathbb{Z}$ to itself $n \in \mathbb{C}$. The map $f : \mathbb{C} \to \mathbb{C}\backslash\{0\}$ is the map taking $z \in \mathbb{C}$ to $e^{2\pi i z} \in \mathbb{C}\backslash\{0\}$.

It is clear that ι is injective and f is surjective (from Complex Analysis). Also, we have $\operatorname{Im}\iota = \mathbb{Z}$ and $\ker f = \mathbb{Z}$ as well (note that the identity of $\mathbb{C}\backslash\{0\}$ is 1, not 0). Therefore, this is an exact sequence.

Exercise 5.11. Given an exact sequence of group homomorphisms:

$$0 \to A \xrightarrow{T} B \xrightarrow{S} C \to 0,$$

(a) If it is given that $C = \{0\}$, what can you say about A and B?

(b) If it is given that $A = \{0\}$, what can you say about B and C?

5.3.2. Mayer-Vietoris Sequences. We talk about exact sequences because there is such a sequence concerning de Rham cohomology groups. This exact sequence, called the Mayer-Vietoris sequence, is particularly useful for computing H^k_{dR} for many manifolds.

The basic setup of a Mayer-Vietoris sequence is a smooth manifold (with or without boundary) which can be expressed a union of two open sets U and V, i.e. $M = U \cup V$. Note that we do not require U and V are disjoint. The intersection $U \cap V$ is a subset of both U and V; and each of U and V is in turn a subset of M. To summarize, we have the following relations of sets:

$$\begin{tikzcd} & U \arrow[dr, "j_U"] & \\ U \cap V \arrow[ur, "i_U"] \arrow[dr, "i_V"'] & & M \\ & V \arrow[ur, "j_V"'] & \end{tikzcd}$$

where i_U, i_V, j_U and j_V are inclusion maps. Each inclusion map, say $j_U : U \to M$, induces a pull-back map $j_U^* : \wedge^k T^*M \to \wedge^k T^*U$ which takes any k-form ω on M, to the k-form $\omega|_U$ restricted on U, i.e. $j_U^*(\omega) = \omega|_U$ for any $\omega \in \wedge^k T^*M$. In terms of local expressions, there is essentially no difference between ω and $\omega|_U$ since U is open. If locally $\omega = \sum_i \omega_i \, du^i$ on M, then $\omega|_U = \sum_i \omega_i \, du^i$ as well. The only difference is the domain: $\omega(p)$ is defined for every $p \in M$, while $\omega|_U (p)$ is defined only when $p \in U$.

To summarize, we have the following diagram:

$$\begin{tikzcd} & U \arrow[dl, "i_U^*"'] & \\ U \cap V & & M \arrow[ul, "j_U^*"'] \arrow[dl, "j_V^*"] \\ & V \arrow[ul, "i_V^*"] & \end{tikzcd}$$

Using the pull-backs of these four inclusions i_U, i_V, j_U and j_V, one can form a sequence of linear maps for each integer k:

$$0 \to \wedge^k T^*M \xrightarrow{j_U^* \oplus j_V^*} \wedge^k T^*U \oplus \wedge^k T^*V \xrightarrow{i_U^* - i_V^*} \wedge^k T^*(U \cap V) \to 0 \tag{5.4}$$

Here, $\wedge^k T^*U \oplus \wedge^k T^*V$ is the direct sum of the vector spaces $\wedge^k T^*U$ and $\wedge^k T^*V$, meaning that:

$$\wedge^k T^*U \oplus \wedge^k T^*V = \{(\omega, \eta) : \omega \in \wedge^k T^*U \text{ and } \eta \in \wedge^k T^*V\}.$$

The map $j_U^* \oplus j_V^* : \wedge^k T^*M \to \wedge^k T^*U \oplus \wedge^k T^*V$ is defined by:

$$(j_U^* \oplus j_V^*)(\omega) = (j_U^*\omega, j_V^*\omega) = (\omega|_U, \omega|_V).$$

The map $\wedge^k T^*U \oplus \wedge^k T^*V \xrightarrow{i_U^* - i_V^*} \wedge^k T^*(U \cap V)$ is given by:

$$(i_U^* - i_V^*)(\omega, \eta) = i_U^*\omega - i_V^*\eta = \omega|_{U\cap V} - \eta|_{U \cap V}.$$

We next show that the sequence (5.4) is exact. Let's first try to understand the image and kernel of each map involved.

Given $(\omega, \eta) \in \ker(i_U^* - i_V^*)$, we will have $\omega|_{U\cap V} = \eta|_{U\cap V}$. Therefore, $\ker(i_U^* - i_V^*)$ consists of pairs (ω, η) where ω and η agree on the intersection $U \cap V$.

Now consider Im $(j_U^* \oplus j_V^*)$, which consists of pairs of the form $(\omega|_U, \omega|_V)$. Certainly, the restrictions of both $\omega|_U$ and $\omega|_V$ on the intersection $U \cap V$ are the same, and hence the pair is inside $\ker(i_U^* - i_V^*)$. Therefore, we have Im $(j_U^* \oplus j_V^*) \subset \ker(i_U^* - i_V^*)$.

In order to show (5.4) is exact, we need further that:

(1) $j_U^* \oplus j_V^*$ is injective;

(2) $i_U^* - i_V^*$ is surjective; and

(3) $\ker(i_U^* - i_V^*) \subset \mathrm{Im}\,(j_U^* \oplus j_V^*)$

We leave (1) as an exercises, and will give the proofs of (2) and (3).

Exercise 5.12. Show that $j_U^* \oplus j_V^*$ is injective in the sequence (5.4).

Proposition 5.15. *Let M be a smooth manifold. Suppose there are two open subsets U and V of M such that $M = U \cup V$, and $U \cap V$ is non-empty, then the sequence of maps* (5.4) *is exact.*

Proof. So far we have proved that $j_U^* \oplus j_V^*$ is injective, and Im $(j_U^* \oplus j_V^*) \subset \ker(i_U^* - i_V^*)$. We next claim that $\ker(i_U^* - i_V^*) \subset \mathrm{Im}\,(j_U^* \oplus j_V^*)$:

Let $(\omega, \eta) \in \ker(i_U^* - i_V^*)$, meaning that ω is a k-form on U, η is a k-form on V, and that $\omega|_{U\cap V} = \eta|_{U\cap V}$. Define a k-form σ on $M = U \cup V$ by:

$$\sigma = \begin{cases} \omega & \text{on } U \\ \eta & \text{on } V \end{cases}$$

Note that σ is well-defined on $U \cap V$ since ω and η agree on $U \cap V$. Then, we have:

$$(\omega, \eta) = (\sigma|_U, \sigma|_V) = (j_U^*\sigma, j_V^*\sigma) = (j_U^* \oplus j_V^*)\sigma \in \mathrm{Im}\,(j_U^* \oplus j_V^*).$$

Since (ω, η) is arbitrary in $\ker(i_U^* - i_V^*)$, this shows:

$$\ker(i_U^* - i_V^*) \subset \mathrm{Im}\,(j_U^* \oplus j_V^*).$$

Finally, we show $i_U^* - i_V^*$ is surjective. Given any k-form $\theta \in \wedge^k T^*(U \cap V)$, we need to find a k-form ω' on U, and a k-form η' on V such that $\omega' - \eta' = \theta$ on $U \cap V$. Let $\{\rho_U, \rho_V\}$ be a partition of unity subordinate to $\{U, V\}$. We define:

$$\omega' = \begin{cases} \rho_V \theta & \text{on } U \cap V \\ 0 & \text{on } U \backslash V \end{cases}$$

Note that ω' is smooth: If $p \in \operatorname{supp} \rho_V \subset V$, then $p \in V$ (which is open) and so $\omega' = \rho_V \theta$ in an open neighborhood of p. Note that ρ_V and θ are smooth at p, so ω' is also smooth at p. On the other hand, if $p \notin \operatorname{supp} \rho_V$, then $\omega' = 0$ in an open neighborhood of p. In particular, ω' is smooth at p.

Similarly, we define:

$$\eta' = \begin{cases} -\rho_U \theta & \text{on } U \cap V \\ 0 & \text{on } V \backslash U \end{cases}$$

which can be shown to be smooth in a similar way.

Then, when restricted to $U \cap V$, we get:

$$\omega'|_{U\cap V} - \eta'|_{U\cap V} = \rho_V \theta + \rho_U \theta = (\rho_V + \rho_U)\,\theta = \theta.$$

In other words, we have $(i_U^* - i_V^*)(\omega', \eta') = \theta$. Since θ is arbitrary, we proved $i_U^* - i_V^*$ is surjective. □

Recall that a pull-back map on k-forms induces a well-defined pull-back map on H^k_{dR}. The sequence of maps (5.4) between space of wedge products induces a sequence of maps between de Rham cohomology groups:

$$0 \to H^k_{\mathrm{dR}}(M) \xrightarrow{j_U^* \oplus j_V^*} H^k_{\mathrm{dR}}(U) \oplus H^k_{\mathrm{dR}}(V) \xrightarrow{i_U^* - i_V^*} H^k_{\mathrm{dR}}(U \cap V) \to 0. \tag{5.5}$$

Here, $j_U^* \oplus j_V^*$ and $i_U^* - i_V^*$ are defined by:

$$(j_U^* \oplus j_V^*)[\omega] = (j_U^*[\omega], j_V^*[\omega]) = ([j_U^*\omega], [j_V^*\omega])$$
$$(i_U^* - i_V^*)\,([\omega], [\eta]) = i_U^*[\omega] - i_V^*[\eta] = [i_U^*\omega] - [i_V^*\eta].$$

However, the sequence (5.5) is *not* exact because $j_U^* \oplus j_V^*$ may not be injective, and $i_U^* - i_V^*$ may not be surjective. For example, take $M = \mathbb{R}^2 \backslash \{(0,0)\}$, and define using polar coordinates the open sets $U = \{re^{i\theta} : r > 0, \theta \in (0, 2\pi)\}$ and $V = \{re^{i\theta} : r > 0, \theta \in (-\pi, \pi)\}$. Then, both U and V are star-shaped and hence both $H^1_{\mathrm{dR}}(U)$ and $H^1_{\mathrm{dR}}(V)$ are trivial. Nonetheless we have exhibited that $H^1_{\mathrm{dR}}(M)$ is non-trivial. The map $j_U^* \oplus j_V^*$ from a non-trivial group to the trivial group can never be injective!

Exercise 5.13. Find an example of M, U and V such that the map $i_U^* - i_V^*$ in (5.5) is not surjective.

Nonetheless, it is still true that $\ker(i_U^* - i_V^*) = \operatorname{Im}(j_U^* \oplus j_V^*)$, and we will verify it in the proof of Mayer-Vietoris Theorem (Theorem 5.16). Mayer-Vietoris Theorem asserts that although (5.5) is not exact in general, but we can connect each short sequence below:

$$H^0_{\mathrm{dR}}(M) \xrightarrow{j_U^* \oplus j_V^*} H^0_{\mathrm{dR}}(U) \oplus H^0_{\mathrm{dR}}(V) \xrightarrow{i_U^* - i_V^*} H^0_{\mathrm{dR}}(U \cap V)$$
$$H^1_{\mathrm{dR}}(M) \xrightarrow{j_U^* \oplus j_V^*} H^1_{\mathrm{dR}}(U) \oplus H^1_{\mathrm{dR}}(V) \xrightarrow{i_U^* - i_V^*} H^1_{\mathrm{dR}}(U \cap V)$$
$$H^2_{\mathrm{dR}}(M) \xrightarrow{j_U^* \oplus j_V^*} H^2_{\mathrm{dR}}(U) \oplus H^2_{\mathrm{dR}}(V) \xrightarrow{i_U^* - i_V^*} H^2_{\mathrm{dR}}(U \cap V)$$
$$\vdots$$

to produce a long exact sequence.

Theorem 5.16 (Mayer-Vietoris Theorem)**.** *Let M be a smooth manifold, and U and V be open sets of M such that $M = U \cup V$. Then, for each $k \geq 0$ there is a homomorphism $\delta : H^k_{\mathrm{dR}}(U \cap V) \to H^{k+1}_{\mathrm{dR}}(M)$ such that the following sequence is exact:*

$$\cdots \xrightarrow{\delta} H^k_{\mathrm{dR}}(M) \xrightarrow{j_U^* \oplus j_V^*} H^k_{\mathrm{dR}}(U) \oplus H^k_{\mathrm{dR}}(V) \xrightarrow{i_U^* - i_V^*} H^k_{\mathrm{dR}}(U \cap V) \xrightarrow{\delta} H^{k+1}_{\mathrm{dR}}(M) \to \cdots$$

This long exact sequence is called the Mayer-Vietoris sequence.

The proof of Theorem 5.16 is purely algebraic. We will learn the proof after looking at some examples.

5.3.3. Using Mayer-Vietoris Sequences.

The Mayer-Vietoris sequence is particularly useful for computing de Rham cohomology groups and Betti numbers using linear algebraic methods. Suppose M can be expressed as a union $U \cup V$ of two open sets, such that the H^k_{dR}'s of U, V and $U \cap V$ can be computed easily, then $H^k_{\mathrm{dR}}(M)$ can be deduced by "playing around" the kernels and images in the Mayer-Vietoris sequence. One useful result in Linear (or Abstract) Algebra is the following:

Theorem 5.17 (First Isomorphism Theorem). *Let $T : V \to W$ be a linear map between two vector spaces V and W. Then, we have:*

$$\operatorname{Im} T \cong V / \ker T.$$

In particular, if V and W are finite dimensional, we have:

$$\dim \ker T + \dim \operatorname{Im} T = \dim V.$$

Proof. Let $\Phi : \operatorname{Im} T \to V / \ker T$ be the map defined by:

$$\Phi(T(v)) = [v]$$

for any $T(v) \in \operatorname{Im} T$. This map is well-defined since if $T(v) = T(w)$ in $\operatorname{Im} T$, then $v - w \in \ker T$, which implies $[v] = [w]$ in the quotient vector space $V / \ker T$. It is easy (hence omitted) to verify that Φ is linear.

Φ is injective since whenever $T(v) \in \ker \Phi$, we have $\Phi(T(v)) = [0]$ which implies $[v] = [0]$ and hence $v \in \ker T$ (i.e. $T(v) = 0$). Also, Φ is surjective since given any $[v] \in V / \ker T$, we have $\Phi(T(v)) = [v]$ by the definition of Φ.

These show Φ is an isomorphism, hence completing the proof. □

Example 5.18. In this example, we use the Mayer-Vietoris sequence to compute $H^1_{\text{dR}}(\mathbb{S}^1)$. Let:

$$M = \mathbb{S}^1, \quad U = M \backslash \{\text{north pole}\}, \quad V = M \backslash \{\text{south pole}\}.$$

Then clearly $M = U \cup V$, and $U \cap V$ consists of two disjoint arcs (each of which deformation retracts to a point). Here are facts which we know and which we haven't yet known:

$$\begin{array}{llll} H^0_{\text{dR}}(M) \cong \mathbb{R} & H^0_{\text{dR}}(U) \cong \mathbb{R} & H^0_{\text{dR}}(V) \cong \mathbb{R} & H^0_{\text{dR}}(U \cap V) \cong \mathbb{R} \oplus \mathbb{R} \\ H^1_{\text{dR}}(M) \text{ unknown} & H^1_{\text{dR}}(U) \cong 0 & H^1_{\text{dR}}(V) \cong 0 & H^1_{\text{dR}}(U \cap V) \cong 0 \end{array}$$

By Theorem 5.16, we know that the following sequence is exact:

$$\cdots \to \underbrace{H^0_{\text{dR}}(U) \oplus H^0_{\text{dR}}(V)}_{\mathbb{R} \oplus \mathbb{R}} \xrightarrow{i_U^* - i_V^*} \underbrace{H^0_{\text{dR}}(U \cap V)}_{\mathbb{R} \oplus \mathbb{R}} \xrightarrow{\delta} \underbrace{H^1_{\text{dR}}(M)}_{?} \xrightarrow{j_U^* \oplus j_V^*} \underbrace{H^1_{\text{dR}}(U) \oplus H^1_{\text{dR}}(V)}_{0}$$

Therefore, δ is surjective.

By First Isomorphism Theorem (Theorem 5.17), we know:

$$H^1_{\text{dR}}(M) = \operatorname{Im} \delta \cong \frac{H^0_{\text{dR}}(U \cap V)}{\ker \delta}.$$

Elements of $H^0_{\text{dR}}(U \cap V)$ are locally constant functions of the form:

$$f_{a,b} = \begin{cases} a & \text{on left arc} \\ b & \text{on right arc} \end{cases}$$

Since the Mayer-Vietoris sequence is exact, we have $\ker \delta = \operatorname{Im}(i_U^* - i_V^*)$. The space $H^0_{\text{dR}}(U)$, $H^0_{\text{dR}}(V)$ and $H^0_{\text{dR}}(U \cap V)$ consist of locally constant functions on U, V and $U \cap V$ respectively, and the maps $i_U^* - i_V^*$ takes constant functions $(k_1, k_2) \in H^0_{\text{dR}}(U) \oplus H^0_{\text{dR}}(V)$ to the constant function $f_{k_1 - k_2, k_1 - k_2}$ on $U \cap V$. Therefore, the first de Rham cohomology group of M is given by:

$$H^1_{\text{dR}}(M) \cong \frac{\{f_{a,b} : a, b \in \mathbb{R}\}}{\{f_{a-b,a-b} : a, b \in \mathbb{R}\}} \cong \frac{\mathbb{R}^2}{\{(x,y) : x = y\}},$$

and hence $b_1(M) = \dim H^1_{\text{dR}}(M) = 2 - 1 = 1$. □

Example 5.19. Let's discuss some consequences of the result proved in the previous example. Recall that $\mathbb{R}^2\backslash\{(0,0)\}$ deformation retracts to $\mathbb{S}^1$. By Theorem 5.13, we know $H^1_{\mathrm{dR}}(\mathbb{R}^2\backslash\{(0,0)\}) \cong H^1_{\mathrm{dR}}(\mathbb{S}^1)$.

This tells us $b_1(\mathbb{R}^2\backslash\{(0,0)\}) = 1$ as well. Recall that the following 1-form:

$$\omega = \frac{-y\,dx + x\,dy}{x^2+y^2}$$

is closed but not exact. The class $[\omega]$ is then trivial in $H^1_{\mathrm{dR}}(\mathbb{R}^2\backslash\{(0,0)\})$. In an one-dimensional vector space, any non-zero vector spans that space. Therefore, we conclude:

$$H^1_{\mathrm{dR}}(\mathbb{R}^2\backslash\{(0,0)\} = \{c[\omega] : c \in \mathbb{R}\}.$$

where ω is defined as in above.

As a result, if ω' is a closed 1-form on $\mathbb{R}^2\backslash\{(0,0)\}$, then we must have

$$[\omega'] = c[\omega]$$

for some $c \in \mathbb{R}$, and so $\omega' = c\omega + df$ for some smooth function $f : \mathbb{R}^2\backslash\{(0,0)\} \to \mathbb{R}$.

Using the language of vector fields, if $\mathsf{V}(x,y) : \mathbb{R}^2\backslash\{(0,0)\} \to \mathbb{R}^2$ is a smooth vector field with $\nabla \times \mathsf{V} = 0$, then there is a constant $c \in \mathbb{R}$ and a smooth function $f : \mathbb{R}^2\backslash\{(0,0)\} \to \mathbb{R}$ such that:

$$\mathsf{V} = c\left(\frac{-y\mathsf{i} + x\mathsf{j}}{x^2+y^2}\right) + \nabla f.$$

□

Exercise 5.14. Let $\mathbb{T}^2$ be the two-dimensional torus. Show that $b_1(\mathbb{T}^2) = 2$.

Exercise 5.15. Show that $b_1(\mathbb{S}^2) = 0$. Based on this result, show that any curl-zero vector field defined on $\mathbb{R}^3\backslash\{(0,0,0)\}$ must be conservative.

One good technique of using the Mayer-Vietoris sequence (as demonstrated in the examples and exercises above) is to consider a segment of the sequence that starts and ends with the trivial space, i.e.

$$0 \to V_1 \to V_2 \to \cdots \to V_n \to 0.$$

If all vector spaces V_i's except one of them are known, then the remaining one (at least its dimension) can be deduced using First Isomorphism Theorem. Below is a useful lemma which is particularly useful for finding the Betti number of a manifold:

Lemma 5.20. *Let the following be an exact sequence of finite dimensional vector spaces:*

$$0 \to V_1 \xrightarrow{T_1} V_2 \xrightarrow{T_2} \cdots \xrightarrow{T_{n-1}} V_n \to 0.$$

Then, we have:

$$\dim V_1 - \dim V_2 + \dim V_3 - \cdots + (-1)^{n-1}\dim V_n = 0$$

Proof. By exact-ness, the map $T_{n-1} : V_{n-1} \to V_n$ is surjective. By First Isomorphism Theorem (Theorem 5.17), we get:

$$V_n = \operatorname{Im} T_{n-1} \cong V_{n-1}/\ker T_{n-1} = V_{n-1}/\operatorname{Im} T_{n-2}.$$

As a result, we have:

$$\dim V_n = \dim V_{n-1} - \dim \operatorname{Im} T_{n-2}.$$

Similarly, apply First Isomorphism Theorem on $T_{n-2} : V_{n-2} \to V_{n-1}$, we get:

$$\dim \operatorname{Im} T_{n-2} = \dim V_{n-2} - \dim \operatorname{Im} T_{n-3},$$

and combine with the previous result, we get:

$$\dim V_n = \dim V_{n-1} - \dim V_{n-2} + \dim \operatorname{Im} T_{n-3}.$$

Proceed similarly as the above, we finally get:

$$\dim V_n = \dim V_{n-1} - \dim V_{n-2} + \ldots + (-1)^n \dim V_1,$$

as desired. $\square$

In Example 5.18 (about computing $H^1_{\mathrm{dR}}(\mathbb{S}^1)$), the following exact sequence was used:

$$0 \to \underbrace{H^0_{\mathrm{dR}}(\mathbb{S}^1)}_{\mathbb{R}} \to \underbrace{H^0_{\mathrm{dR}}(U) \oplus H^0_{\mathrm{dR}}(V)}_{\mathbb{R}\oplus\mathbb{R}} \to \underbrace{H^0_{\mathrm{dR}}(U \cap V)}_{\mathbb{R}\oplus\mathbb{R}} \to \underbrace{H^1_{\mathrm{dR}}(\mathbb{S}^1)}_{?} \to \underbrace{H^1_{\mathrm{dR}}(U) \oplus H^1_{\mathrm{dR}}(V)}_{0}$$

Using Lemma 5.20, the dimension of $H^1_{\mathrm{dR}}(\mathbb{S}^1)$ can be computed easily:

$$\dim \mathbb{R} - \dim \mathbb{R} \oplus \mathbb{R} + \dim \mathbb{R} \oplus \mathbb{R} - \dim H^1_{\mathrm{dR}}(\mathbb{S}^1) = 0$$

which implies $\dim H^1_{\mathrm{dR}}(\mathbb{S}^1) = 1$ (or equivalently, $b_1(\mathbb{S}^1) = 1$). Although this method does not give a precise description of $H^1_{\mathrm{dR}}(\mathbb{S}^1)$ in terms of inclusion maps, it is no doubt much easier to adopt.

In the forthcoming examples, we will assume the following facts stated below (which we have only proved the case $k = 1$):

- $H^k_{\mathrm{dR}}(U) = 0$, where $k \geq 1$, for any star-shaped region $U \subset \mathbb{R}^n$.
- If Σ is a deformation retract of M, then $H^k_{\mathrm{dR}}(\Sigma) \cong H^k_{\mathrm{dR}}(M)$ for any $k \geq 1$.

Example 5.21. Consider $\mathbb{R}^2 \backslash \{p_1, \ldots, p_n\}$ where $p_1, \ldots, p_n$ are n distinct points in $\mathbb{R}^2$. We want to find b_1 of this open set.

Define $U = \mathbb{R}^2 \backslash \{p_1, \ldots, p_{n-1}\}$, $V = \mathbb{R}^2 \backslash \{p_n\}$, then $U \cup V = \mathbb{R}^2$ and $U \cap V = \mathbb{R}^2 \backslash \{p_1, \ldots, p_n\}$. Consider the Mayer-Vietoris sequence:

$$\underbrace{H^1_{\mathrm{dR}}(U \cup V)}_{0} \to H^1_{\mathrm{dR}}(U) \oplus H^1_{\mathrm{dR}}(V) \to H^1_{\mathrm{dR}}(U \cap V) \to \underbrace{H^2_{\mathrm{dR}}(U \cup V)}_{0}.$$

Using Lemma 5.20, we know:

$$\dim H^1_{\mathrm{dR}}(U) \oplus H^1_{\mathrm{dR}}(V) - \dim H^1_{\mathrm{dR}}(U \cap V) = 0$$

We have already figured out that $\dim H^1_{\mathrm{dR}}(V) = 1$. Therefore, we get:

$$\dim H^1_{\mathrm{dR}}(\mathbb{R}^2 \backslash \{p_1, \ldots, p_n\}) = \dim H^1_{\mathrm{dR}}(\mathbb{R}^2 \backslash \{p_1, \ldots, p_{n-1}\}) + 1.$$

By induction, we conclude:

$$b_1(\mathbb{R}^2 \backslash \{p_1, \ldots, p_n\}) = \dim H^1_{\mathrm{dR}}(\mathbb{R}^2 \backslash \{p_1, \ldots, p_n\}) = n.$$

$\square$

Example 5.22. Consider the n-sphere S^n (where $n \geq 2$). It can be written as $U \cup V$ where $U := S^n \backslash \{\text{north pole}\}$ and $V := S^n \backslash \{\text{south pole}\}$. Using stereographic projections, one can show both U and V are diffeomorphic to $\mathbb{R}^n$. Furthermore, $U \cap V$ is diffeomorphic to $\mathbb{R}^n \backslash \{0\}$, which deformation retracts to S^{n-1}. Hence $H^k_{\mathrm{dR}}(S^{n-1}) = H^k_{\mathrm{dR}}(U \cap V)$ for any k.

Now consider the Mayer-Vietoris sequence with these U and V, we have for each $k \geq 2$ an exact sequence:

$$\underbrace{H^{k-1}_{\mathrm{dR}}(U) \oplus H^{k-1}_{\mathrm{dR}}(V)}_{0} \to H^{k-1}_{\mathrm{dR}}(U \cap V) \to H^k_{\mathrm{dR}}(S^n) \to \underbrace{H^k_{\mathrm{dR}}(U) \oplus H^k_{\mathrm{dR}}(V)}_{0}.$$

This shows $H^{k-1}_{\mathrm{dR}}(S^{n-1}) \cong H^k_{\mathrm{dR}}(S^n)$ for any $k \geq 2$. By induction, we conclude that $H^n_{\mathrm{dR}}(S^n) \cong H^1_{\mathrm{dR}}(\mathbb{S}^1) \cong \mathbb{R}$ for any $n \geq 2$. □

5.3.4. Proof of Mayer-Vietoris Theorem. To end this chapter (and this course), we present the proof of the Mayer-Vietoris's Theorem (Theorem 5.16). As mentioned before, the proof is purely algebraic. The key ingredient of the proof applies to many other kinds of cohomologies as well (de Rham cohomology is only one kind of many types of cohomology).

For simplicity, we denote:

$$X^k := \wedge^k T^*M \qquad Y^k := \wedge^k T^*U \oplus \wedge^k T^*V \qquad Z^k := \wedge^k T^*(U \cap V)$$
$$H^k(X) := H^k_{\mathrm{dR}}(M) \qquad H^k(Y) := H^k_{\mathrm{dR}}(U) \oplus H^k_{\mathrm{dR}}(V) \qquad H^k(Z) := H^k_{\mathrm{dR}}(U \cap V)$$

Furthermore, we denote the pull-back maps $i_U^* - i_V^*$ and $j_U^* \oplus j_V^*$ by simply i and j respectively. We then have the following commutative diagram between all these X, Y and Z:

$$\begin{tikzcd}
0 \arrow[r] & X^k \arrow[r, "j"] \arrow[d, "d"] & Y^k \arrow[r, "i"] \arrow[d, "d"] & Z^k \arrow[r] \arrow[d, "d"] & 0 \\
0 \arrow[r] & X^{k+1} \arrow[r, "j"] \arrow[d, "d"] & Y^{k+1} \arrow[r, "i"] \arrow[d, "d"] & Z^{k+1} \arrow[r] \arrow[d, "d"] & 0 \\
0 \arrow[r] & X^{k+2} \arrow[r, "j"] & Y^{k+2} \arrow[r, "i"] & Z^{k+2} \arrow[r] & 0
\end{tikzcd}$$

The maps in the diagram commute because the exterior derivative d commute with any pull-back map. The map $d : Y^k \to Y^{k+1}$ takes (ω, η) to $(d\omega, d\eta)$.

To give a proof of the Mayer-Vietoris Theorem, we first need to construct a linear map $\delta : H^k_{\mathrm{dR}}(Z) \to H^{k+1}_{\mathrm{dR}}(Z)$. Then, we need to check that the connected sequence:

$$\cdots \xrightarrow{i} H^k(Z) \xrightarrow{\delta} H^{k+1}(X) \xrightarrow{j} H^{k+1}(Y) \xrightarrow{i} H^{k+1}(Z) \xrightarrow{\delta} \cdots$$

is exact. Most arguments involved are done by "chasing the commutative diagram".

<u>**Step 1**: Construction of $H^k(Z) \xrightarrow{\delta} H^{k+1}(X)$</u>

Let $[\theta] \in H^k(Z)$, where $\theta \in Z^k$ is a closed k-form on $U \cap V$. Recall from Proposition 5.15 that the sequence

$$0 \to X^k \xrightarrow{j} Y^k \xrightarrow{i} Z^k \to 0$$

is exact, and in particular i is surjective. As a result, there exists $\omega \in Y^k$ such that $i(\omega) = \theta$.

From the commutative diagram, we know $id\omega = di\omega = d\theta = 0$, and hence $d\omega \in \ker i$. By exact-ness, $\operatorname{Im} j = \ker i$ and so there exists $\eta \in X^{k+1}$ such that $j(\eta) = d\omega$.

Next we argue that such η must be closed: since $j(d\eta) = d(j\eta) = d(d\omega) = 0$, and j is injective by exact-ness. We must have $d\eta = 0$, and so η represents a class in $H^{k+1}(X)$. To summarize, given $[\theta] \in H^k(Z)$, ω and η are elements such that

$$i(\omega) = \theta \quad \text{and} \quad j(\eta) = d\omega.$$

We then define $\delta[\theta] := [\eta] \in H^{k+1}(X)$.

<u>**Step 2:** Verify that δ is a well-defined map</u>

Suppose $[\theta] = [\theta']$ in $H^k_{\mathrm{dR}}(Z)$. Let $\omega' \in Y^k$ and $\eta' \in X^{k+1}$ be the corresponding elements associated with θ', i.e.

$$i(\omega') = \theta' \quad \text{and} \quad j(\eta') = d\omega'.$$

We need to show $[\eta] = [\eta']$ in $H^{k+1}(X)$.

From $[\theta] = [\theta']$, there exists a $(k-1)$-form β in Z^{k-1} such that $\theta - \theta' = d\beta$, which implies:

$$i(\omega - \omega') = \theta - \theta' = d\beta.$$

By surjectivity of $i : Y^{k-1} \to Z^{k-1}$, there exists $\alpha \in Y^{k-1}$ such that $i\alpha = \beta$. Then we get:

$$i(\omega - \omega') = d(i\alpha) = id\alpha$$

which implies $(\omega - \omega') - d\alpha \in \ker i$.

By exact-ness, $\ker i = \operatorname{Im} j$ and so there exists $\gamma \in X^k$ such that

$$j\gamma = (\omega - \omega') - d\alpha.$$

Differentiating both sides, we arrive at:

$$dj\gamma = d(\omega - \omega') - d^2\alpha = j(\eta - \eta').$$

Therefore, $jd\gamma = dj\gamma = j(\eta - \eta')$, and by injectivity of j, we get:

$$\eta - \eta' = d\gamma$$

and so $[\eta] = [\eta']$ in $H^{k+1}(X)$.

<u>**Step 3:** Verify that δ is a linear map</u>

We leave this step as an exercise for readers.

<u>**Step 4:** Check that $H^k(Y) \xrightarrow{i} H^k(Z) \xrightarrow{\delta} H^{k+1}(X)$ is exact</u>

To prove $\operatorname{Im} i \subset \ker \delta$, we take an arbitrary $[\theta] \in \operatorname{Im} i \subset H^k(Z)$, there is $[\omega] \in H^k(Y)$ such that $[\theta] = i[\omega]$, we will show $\delta[\theta] = 0$. Recall that $\delta[i\omega]$ is the element $[\eta]$ in $H^{k+1}(X)$ such that $j\eta = d\omega$. Now that ω is closed, the injectivity of j implies $\eta = 0$. Therefore, $\delta[\theta] = \delta[i\omega] = [0]$, proving $[\theta] \in \ker \delta$.

Next we show $\ker \delta \subset \operatorname{Im} i$. Suppose $[\theta] \in \ker \delta$, and let ω and η be the forms such that $i(\omega) = \theta$ and $j(\eta) = d\omega$. Then $[\eta] = \delta[\theta] = [0]$, so there exists $\gamma \in X^{k-1}$ such that $\eta = d\gamma$, which implies $j(d\gamma) = d\omega$, and so $\omega - j\gamma$ is closed. By exact-ness, $i(j\gamma) = 0$, and so:

$$\theta = i(\omega) = i(\omega - j\gamma).$$

For $\omega - j\gamma$ being closed, we conclude $[\theta] = i[\omega - j\gamma] \in \operatorname{Im} i$ in $H^k(Z)$.

<u>**Step 5:** Check that $H^k(Z) \xrightarrow{\delta} H^{k+1}(X) \xrightarrow{j} H^{k+1}(Y)$ is exact</u>

First show $\operatorname{Im} \delta \subset \ker j$. Let $[\theta] \in H^{k+1}(Z)$, then $\delta[\theta] = [\eta]$ where

$$i(\omega) = \theta \quad \text{and } j(\eta) = d\omega.$$

As a result, $j\delta[\theta] = j[\eta] = [d\omega] = [0]$. This shows $\delta[\theta] \in \ker j$.

Next we show $\ker j \subset \operatorname{Im} \delta$. Let $j[\omega] = [0]$, then $j\omega = d\alpha$ for some $\alpha \in Y^k$. Since:

$$i(\alpha) = i\alpha \quad \text{and} \quad j(\omega) = d\alpha$$

We conclude $\delta[i\alpha] = [\omega]$, or in other words, $[\omega] \in \operatorname{Im} \delta$.

<u>**Step 6:** Check that $H^{k+1}(X) \xrightarrow{j} H^{k+1}(Y) \xrightarrow{i} H^{k+1}(Z)$ is exact</u>

The inclusion $\operatorname{Im} j \subset \ker i$ follows from the fact that $i(j\eta) = 0$ for any closed $\eta \in X^{k+1}$, and hence $ij[\eta] = [0]$. Finally, we show $\ker i \subset \operatorname{Im} j$: suppose $[\omega] \in \ker i$ so that $i\omega = d\beta$ for some $\beta \in Z^k$. By surjectivity of $i : Y^k \to Z^k$, there exists $\alpha \in Y^k$ such that $\beta = i\alpha$. As a result, we get:

$$i\omega = di\alpha = id\alpha \quad \Longrightarrow \quad \omega - d\alpha \in \ker i.$$

Since $\ker i = \operatorname{Im} j$ on the level of $X^{k+1} \to Y^{k+1} \to Z^{k+1}$, there exists $\gamma \in X^{k+1}$ such that $j\gamma = \omega - d\alpha$. One can easily show γ is closed by injectivity of j:

$$jd\gamma = dj\gamma = d(\omega - d\alpha) = 0 \quad \Longrightarrow \quad d\gamma = 0$$

and so $[\gamma] \in H^{k+1}(X)$. Finally, we conclude:

$$j[\gamma] = [\omega - d\alpha] = [\omega]$$

and so $[\omega] \in \operatorname{Im} j$.

*** End of the proof of the Mayer-Vietoris Theorem ***
**** End of MATH 4033 ****
***** I hope you enjoy it. *****