Chapter 4

Generalized Stokes' Theorem

"It is very difficult for us, placed as we have been from earliest childhood in a condition of training, to say what would have been our feelings had such training never taken place."

Sir George Stokes, 1st Baronet

4.1. Manifolds with Boundary

We have seen in the Chapter 3 that Green's, Stokes' and Divergence Theorem in Multivariable Calculus can be unified together using the language of differential forms. In this chapter, we will generalize Stokes' Theorem to higher dimensional and abstract manifolds.

These classic theorems and their generalizations concern about an integral over a manifold with an integral over its boundary. In this section, we will first rigorously define the notion of a *boundary* for abstract manifolds. Heuristically, an *interior* point of a manifold locally looks like a ball in Euclidean space, whereas a *boundary* point locally looks like an upper-half space.

4.1.1. Smooth Functions on Upper-Half Spaces. From now on, we denote $\mathbb{R}^n_+ := \{(u_1, \ldots, u_n) \in \mathbb{R}^n : u_n \geq 0\}$ which is the upper-half space of $\mathbb{R}^n$. Under the subspace topology, we say a subset $V \subset \mathbb{R}^n_+$ is *open in* $\mathbb{R}^n_+$ if there exists a set $\widetilde{V} \subset \mathbb{R}^n$ open in $\mathbb{R}^n$ such that $V = \widetilde{V} \cap \mathbb{R}^n_+$. It is intuitively clear that if $V \subset \mathbb{R}^n_+$ is disjoint from the subspace $\{u_n = 0\}$ of $\mathbb{R}^n$, then V is open in $\mathbb{R}^n_+$ if and only if V is open in $\mathbb{R}^n$.

Now consider a set $V \subset \mathbb{R}^n_+$ which is open in $\mathbb{R}^n_+$ and that $V \cap \{u_n = 0\} \neq \emptyset$. We need to first develop a notion of differentiability for functions such an V as their domain. Given a vector-valued function $\mathsf{G} : V \to \mathbb{R}^m$, then near a point $\mathsf{u} \in V \cap \{u_n = 0\}$, we can only approach u from one side only, namely from directions with positive u_n-coordinates. The usual definition of differentiability does not apply at such a point, so we define:

Definition 4.1 (Functions of Class C^k on $\mathbb{R}^n_+$)**.** Let $V \subset \mathbb{R}^n_+$ be open in $\mathbb{R}^n_+$ and that $V \cap \{u_n = 0\} \neq \emptyset$. Consider a vector-valued function $\mathsf{G} : V \to \mathbb{R}^m$. We say G is C^k (resp. smooth) at $\mathsf{u} \in V \cap \{u_n = 0\}$ if there exists a C^k (resp. smooth) local extension $\widetilde{\mathsf{G}} : B_\varepsilon(\mathsf{u}) \to \mathbb{R}^m$ such that $\widetilde{\mathsf{G}}(\mathsf{y}) = \mathsf{G}(\mathsf{y})$ for any $\mathsf{y} \in B_\varepsilon(\mathsf{u}) \cap V$. Here $B_\varepsilon(\mathsf{u}) \subset \mathbb{R}^n$ refers to an open ball in $\mathbb{R}^n$.

If G is C^k (resp. smooth) at every $\mathsf{u} \in V$ (including those points with $u_n > 0$), then we say G is C^k (resp. smooth) on V.

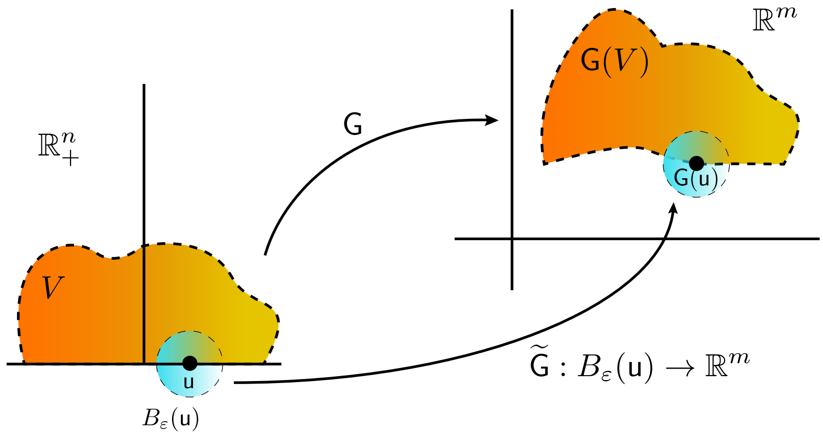

Figure 4.1. G is C^k at u if there exists a local extension $\widetilde{\mathsf{G}}$ near u.

Example 4.2. Let $V = \{(x, y) : y \geq 0 \text{ and } x^2 + y^2 < 1\}$, which is an open set in $\mathbb{R}^2_+$ since $V = \underbrace{\{(x, y) : x^2 + y^2 < 1\}}_{\text{open in } \mathbb{R}^2} \cap \mathbb{R}^2_+$. Then $f(x, y) : V \to \mathbb{R}$ defined by $f(x, y) = \sqrt{1 - x^2 - y^2}$ is a smooth function on V since $\sqrt{1 - x^2 - y^2}$ is smoothly on the whole ball $x^2 + y^2 < 1$.

However, the function $g : V \to \mathbb{R}$ defined by $g(x, y) = \sqrt{y}$ is not smooth at every point on the y-axis because $\frac{\partial g}{\partial y} \to \infty$ as $y \to 0^+$. Any extension $\widetilde{g}$ of g will agree with g on the upper-half plane, and hence will also be true that $\frac{\partial \widetilde{g}}{\partial y} \to \infty$ as $y \to 0^+$, which is sufficient to argue that such $\widetilde{g}$ is not smooth. □

Exercise 4.1. Consider $f : \mathbb{R}^2_+ \to \mathbb{R}$ defined by $f(x, y) = |x|$. Is f smooth on $\mathbb{R}^2_+$? If not, at which point(s) in $\mathbb{R}^2_+$ is f not smooth? Do the same for $g : \mathbb{R}^2_+ \to \mathbb{R}$ defined by $g(x, y) = |y|$.

4.1.2. Boundary of Manifolds. After understanding the definition of a smooth function when defined on subsets of the upper-half space, we are ready to introduce the notion of manifolds with boundary:

Definition 4.3 (Manifolds with Boundary)**.** We say M is a smooth *manifold with boundary* if there exist two families of local parametrizations $\mathsf{F}_\alpha : \mathcal{U}_\alpha \to M$ where $\mathcal{U}_\alpha$ is open in $\mathbb{R}^n$, and $\mathsf{G}_\beta : \mathcal{V}_\beta \to M$ where $\mathcal{V}_\beta$ is open in $\mathbb{R}^n_+$ such that every F_α and G_β is a homeomorphism between its domain and image, and that the transition functions of all types:

$$\mathsf{F}_\alpha^{-1} \circ \mathsf{F}_{\alpha'} \qquad \mathsf{F}_\alpha^{-1} \circ \mathsf{G}_\beta \qquad \mathsf{G}_\beta^{-1} \circ \mathsf{G}_{\beta'} \qquad \mathsf{G}_\beta^{-1} \circ \mathsf{F}_\alpha$$

are smooth on the overlapping domain for any α, α', β and β'.

Moreover, we denote and define the boundary of M by:

$$\partial M := \bigcup_\beta \{\mathsf{G}_\beta(u_1, \ldots, u_{n-1}, 0) : (u_1, \ldots, u_{n-1}, 0) \in \mathcal{V}_\beta\}.$$

Remark 4.4. In this course, we will call these F_α's to be local parametrizations of *interior type*, and these G_β's to be local parametrizations of *boundary type*. □

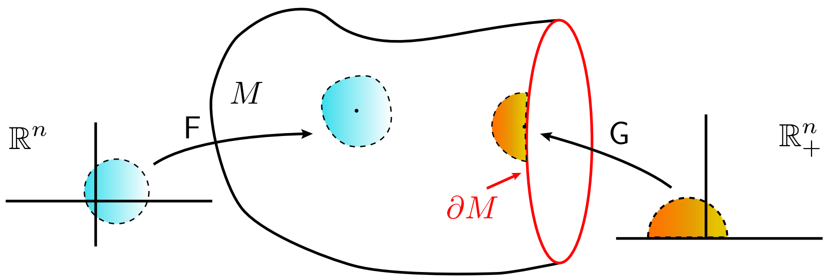

Figure 4.2. A manifold with boundary

Example 4.5. Consider the solid ball $\mathbb{B}^2 := \{\mathsf{x} \in \mathbb{R}^2 : |\mathsf{x}| \leq 1\}$. It can be locally parametrized using polar coordinates by:

$$\mathsf{G} : (0, 2\pi) \times [0, 1) \to \mathbb{B}^2$$
$$\mathsf{G}(\theta, r) := (1 - r)(\cos\theta, \sin\theta)$$

Note that the domain of G can be regarded as a subset

$$\mathcal{V} := \{(\theta, r) : \theta \in (0, 2\pi) \text{ and } 0 \leq r < 1\} \subset \mathbb{R}^2_+.$$

Here we used $1 - r$ instead of r so that the boundary of $\mathbb{B}^2$ has zero r-coordinate, and the interior of $\mathbb{B}^2$ has positive r-coordinate.

Note that the image of G does not cover the whole solid ball $\mathbb{B}^2$. Precisely, the image of G is $\mathbb{B}^2 \backslash \{\text{non-negative } x\text{-axis}\}$. In order to complete the proof that $\mathbb{B}^2$ is a manifold with boundary, we cover $\mathbb{B}^2$ by two more local parametrizations:

$$\widetilde{\mathsf{G}} : (-\pi, \pi) \times [0, 1) \to \mathbb{B}^2$$
$$\widetilde{\mathsf{G}}(\theta, r) := (1 - r)(\cos\theta, \sin\theta)$$

and also the inclusion map $\iota : \{\mathsf{u} \in \mathbb{R}^2 : |\mathsf{u}| < 1\} \to \mathbb{B}^2$. We need to show that the transition maps are smooth. There are six possible transition maps:

$$\widetilde{\mathsf{G}}^{-1} \circ \mathsf{G}, \quad \mathsf{G}^{-1} \circ \widetilde{\mathsf{G}}, \quad \iota^{-1} \circ \mathsf{G}, \quad \iota^{-1} \circ \widetilde{\mathsf{G}}, \quad \mathsf{G}^{-1} \circ \iota, \quad \text{and} \quad \widetilde{\mathsf{G}}^{-1} \circ \iota.$$

The first one is given by (we leave it as an exercise for computing these transition maps):

$$\widetilde{\mathsf{G}}^{-1} \circ \mathsf{G} : ((0,\pi) \cup (\pi, 2\pi)) \times [0,1) \to ((-\pi, 0) \cup (0,\pi)) \times [0,1)$$

$$\widetilde{\mathsf{G}}^{-1} \circ \mathsf{G}(\theta, r) = \begin{cases} (\theta, r) & \text{if } \theta \in (0,\pi) \\ (\theta - 2\pi, r) & \text{if } \theta \in (\pi, 2\pi) \end{cases}$$

which can be smoothly extended to the domain $((0,\pi) \cup (\pi, 2\pi)) \times (-1,1)$. Therefore, $\widetilde{\mathsf{G}}^{-1} \circ \mathsf{G}$ is smooth. The second transition map $\mathsf{G}^{-1} \circ \widetilde{\mathsf{G}}$ can be computed and verified to be smooth in a similar way.

For $\iota^{-1} \circ \mathsf{G}$, by examining the overlap part of ι and G on $\mathbb{B}^2$, we see that the domain of the transition map is an open set $(0, 2\pi) \times (0,1)$ in $\mathbb{R}^2$. On this domain, $\iota^{-1} \circ \mathsf{G}$ is essentially G, which is clearly smooth. Similar for $\iota^{-1} \circ \widetilde{\mathsf{G}}$.

To show $\mathsf{G}^{-1} \circ \iota$ is smooth, we use the Inverse Function Theorem. The domain of $\iota^{-1} \circ \mathsf{G}$ is $(0,2\pi) \times (0,1)$. By writing $(x,y) = \iota^{-1} \circ \mathsf{G}(\theta, r) = (1-r)(\cos\theta, \sin\theta)$, we check that on the domain of $\iota^{-1} \circ \mathsf{G}$, we have:

$$\det \frac{\partial(x,y)}{\partial(\theta,r)} = 1 - r \neq 0.$$

Therefore, the inverse $\mathsf{G}^{-1} \circ \iota$ is smooth. Similar for $\widetilde{\mathsf{G}}^{-1} \circ \iota$.

Combining all of the above verifications, we conclude that $\mathbb{B}^2$ is a 2-dimensional manifold with boundary. The boundary $\partial\mathbb{B}^2$ is given by points with zero r-coordinates, namely the unit circle $\{|\mathsf{x}| = 1\}$. □

Exercise 4.2. Compute all transition maps

$$\widetilde{\mathsf{G}}^{-1} \circ \mathsf{G}, \quad \mathsf{G}^{-1} \circ \widetilde{\mathsf{G}}, \quad \iota^{-1} \circ \mathsf{G}, \quad \iota^{-1} \circ \widetilde{\mathsf{G}}, \quad \mathsf{G}^{-1} \circ \iota, \quad \text{and} \quad \widetilde{\mathsf{G}}^{-1} \circ \iota$$

in Example 4.5. Indicate clearly their domains, and verify that they are smooth on their domains.

Exercise 4.3. Let $f : \mathbb{R}^n \to \mathbb{R}$ be a smooth scalar function. The region in $\mathbb{R}^{n+1}$ above the graph of f is given by:

$$\Gamma_f := \{(u_1, \ldots, u_{n+1}) \in \mathbb{R}^{n+1} : u_{n+1} \geq f(u_1, \ldots, u_n)\}.$$

Show that Γ_f is an n-dimensional manifold with boundary, and the boundary $\partial\Gamma_f$ is the graph of f in $\mathbb{R}^{n+1}$.

Exercise 4.4. Show that ∂M (assumed non-empty) of any n-dimensional manifold M is an $(n-1)$-dimensional manifold without boundary.

From the above example and exercise, we see that verifying a set is a manifold with boundary may be cumbersome. The following proposition provides us with a very efficient way to do so.

Proposition 4.6. *Let $f : M^m \to \mathbb{R}$ be a smooth function from a smooth manifold M. Suppose $c \in \mathbb{R}$ such that the set $\Sigma := f^{-1}([c,\infty))$ is non-empty and that f is a submersion at any $p \in f^{-1}(c)$, then the set Σ is an m-dimensional manifold with boundary. The boundary $\partial\Sigma$ is given by $f^{-1}(c)$.*

Proof. We need to construct local parametrizations for the set Σ. Given any point $p \in \Sigma$, then by the definition of Σ, we have $f(p) > c$ or $f(p) = c$.

For the former case $f(p) > c$, we are going to show that near p there is a local parametrization of Σ of interior type. Regarding p as a point in the manifold M, there exists a smooth local parametrization $\mathsf{F} : \mathcal{U} \subset \mathbb{R}^n \to M$ of M covering p. We argue that such a local parametrization of M induces naturally a local parametrization of Σ near p. Note that f is continuous and so $f^{-1}(c, \infty)$ is an open set of M containing p. Denote $\mathcal{O} = f^{-1}(c, \infty)$, then F restricted to $\mathcal{U} \cap \mathsf{F}^{-1}(\mathcal{O})$ will have its image in $\mathcal{O} \subset \Sigma$, and so is a local parametrization of Σ near p.

For the later case $f(p) = c$, we are going to show that near p there is a local parametrization of Σ of boundary type. Since f is a submersion at p, by the Submersion Theorem (Theorem 2.48) there exist a local parametrization $\mathsf{G} : \widetilde{\mathcal{U}} \to M$ of M near p, and a local parametrization H of $\mathbb{R}$ near c such that $\mathsf{G}(0) = p$ and $\mathsf{H}(0) = c$, and:

$$\mathsf{H}^{-1} \circ f \circ \mathsf{G}(u_1, \ldots, u_m) = u_m.$$

Without loss of generality, we assume that H is an increasing function near 0. We argue that by restricting the domain of G to $\mathcal{U} \cap \{u_m \geq 0\}$, which is an open set in $\mathbb{R}^m_+$, the restricted G is a boundary-type local parametrization of Σ near p. To argue this, we note that:

$$f(\mathsf{G}(u_1, \ldots, u_m)) = \mathsf{H}(u_m) \geq \mathsf{H}(0) = c \quad \text{whenever } u_m \geq 0.$$

Therefore, $\mathsf{G}(u_1, \ldots, u_m) \in f^{-1}([c, \infty)) = \Sigma$ whenever $u_m \geq 0$, and so G (when restricted to $\mathcal{U} \cap \{u_m \geq 0\}$) is a local parametrization of Σ.

Since all local parametrizations F and G of Σ constructed above are induced from local parametrizations of M (whether it is of interior or boundary type), their transition maps are all smooth. This shows Σ is an m-dimensional manifold with boundary. To identify the boundary, we note that for any boundary-type local parametrization G constructed above, we have:

$$\mathsf{H}^{-1} \circ f \circ \mathsf{G}(u_1, \ldots, u_{m-1}, 0) = 0$$

and so $f(\mathsf{G}(u_1, \ldots, u_{m-1})) = \mathsf{H}(0) = c$, and therefore:

$$\mathsf{G}(u_1, \ldots, u_{m-1}, 0) \in f^{-1}(c).$$

This show $\partial\Sigma \subset f^{-1}(c)$. The other inclusion $f^{-1}(c) \subset \partial\Sigma$ follows from the fact that for any $p \in f^{-1}(c)$, the boundary-type local parametrization G has the property that $\mathsf{G}(0) = p$ (and hence $p = \mathsf{G}(0, \ldots, 0, 0) \in \partial\Sigma$). □

Remark 4.7. It is worthwhile to note that the above proof only requires that f is a submersion at any $p \in f^{-1}(c)$, and we do not require that it is a submersion at any $p \in \Sigma = f^{-1}([c, \infty))$. Furthermore, the codomain of f is $\mathbb{R}$ which has dimension 1, hence f is a submersion at p if and only if the tangent map $(f_*)_p$ at p is non-zero – and so it is very easy to verify this condition. □

With the help of Proposition 4.6, one can show many sets are manifolds with boundary by picking a suitable submersion f.

Example 4.8. The n-dimensional ball $\mathbb{B}^n = \{\mathsf{x} \in \mathbb{R}^n : |\mathsf{x}| \leq 1\}$ is an n-manifold with boundary. To argue this, let $f : \mathbb{R}^n \to \mathbb{R}$ be the function:

$$f(\mathsf{x}) = 1 - |\mathsf{x}|^2.$$

Then $\mathbb{B}^n = f^{-1}([0, \infty))$.

The tangent map f_* is represented by the matrix:

$$[f_*] = \left[\frac{\partial f}{\partial x_1}, \; \cdots, \; \frac{\partial f}{\partial x_n}\right] = -2\,[x_1, \; \cdots, \; x_n]$$

which is surjective if and only if $(x_1, \ldots, x_n) \neq (0, \ldots, 0)$. For any $\mathsf{x} \in f^{-1}(0)$, we have $|\mathsf{x}|^2 = 1$ and so in particular $\mathsf{x} \neq 0$. Therefore, f is a submersion at every $\mathsf{x} \in f^{-1}(0)$. By Proposition 4.6, we proved $\mathbb{B}^n = f^{-1}([0, \infty))$ is an n-dimensional manifold with boundary, and the boundary is $f^{-1}(0) = \{\mathsf{x} \in \mathbb{R}^n : |\mathsf{x}| = 1\}$, i.e. the unit circle. □

Exercise 4.5. Suppose $f : M^m \to \mathbb{R}$ is a smooth function defined on a smooth manifold M. Suppose $a, b \in \mathbb{R}$ such that $\Sigma := f^{-1}([a, b])$ is non-empty, and that f is a submersion at any $p \in f^{-1}(a)$ and any $q \in f^{-1}(b)$. Show that Σ is an m-manifold with boundary, and $\partial\Sigma = f^{-1}(a) \cup f^{-1}(b)$.

4.1.3. Tangent Spaces at Boundary Points. On a manifold M^n without boundary, the tangent space T_pM at p is the span of partial differential operators $\left\{ \left.\frac{\partial}{\partial u_i}\right|_p \right\}_{i=1}^n$, where $(u_1, \ldots, u_n)$ are local coordinates of a parametrization $\mathsf{F}(u_1, \ldots, u_n)$ near p.

Now on a manifold M^n with boundary, near any boundary point $p \in \partial M^n$ there exists a local parametrization $\mathsf{G}(u_1, \ldots, u_n) : \mathcal{V} \subset \mathbb{R}^n_+ \to M$ of boundary type. Although G is only defined when $u_n \geq 0$, we *still* define T_pM to be the span of $\left\{ \left.\frac{\partial}{\partial u_i}\right|_p \right\}_{i=1}^n$. Although such a definition of T_pM (when $p \in \partial M$) is a bit counter-intuitive, the perk is that T_pM is still a vector space. Given a vector $V \in T_pM$ with coefficients:

$$V = \sum_{i=1}^n V^i \left.\frac{\partial}{\partial u_i}\right|_p.$$

We say that V is *inward-pointing* if $V^n > 0$; and *outward-pointing* if $V^n < 0$.

Furthermore, the tangent space $T_p(\partial M)$ of the boundary manifold ∂M at p can be regarded as a subspace of T_pM:

$$T_p(\partial M) = \operatorname{span}\left\{ \left.\frac{\partial}{\partial u_i}\right|_p \right\}_{i=1}^{n-1} \subset T_pM.$$

4.2. Orientability

In Multivariable Calculus, we learned (or was told) that Stokes' Theorem requires the surface to be orientable, meaning that the unit normal vector $\hat{\mathsf{n}}$ varies continuously on the surface. The Möbius strip is an example of *non-orientable* surface.

Now we are talking about abstract manifolds which may not sit inside any Euclidean space, and so it does not make sense to define *normal vectors* to the manifold. Even when the manifold M is a subset of $\mathbb{R}^n$, if the dimension of the manifold is $\dim M \leq n-2$, the manifold does not have a unique normal vector direction. As such, in order to generalize the notion of orientability of abstract manifolds, we need to seek a reasonable definition without using normal vectors.

In this section, we first show that for hypersurfaces M^n in $\mathbb{R}^{n+1}$, the notion of orientability using normal vectors is equivalent to another notion using transition maps. Then, we extend the notion of orientability to abstract manifolds using transition maps.

4.2.1. Orientable Hypersurfaces. To begin, we first state the definition of orientable hypersurfaces in $\mathbb{R}^{n+1}$:

Definition 4.9 (Orientable Hypersurfaces)**.** A regular hypersurface M^n in $\mathbb{R}^{n+1}$ is said to be *orientable* if there exists a continuous unit normal vector $\hat{\mathsf{n}}$ defined on the whole M^n

Let's explore the above definition a bit in the easy case $n = 2$. Given a regular surface M^2 in $\mathbb{R}^3$ with a local parametrization $(x, y, z) = \mathsf{F}(u_1, u_2) : \mathcal{U} \to M$, one can find a normal vector to the surface by taking cross product:

$$\frac{\partial \mathsf{F}}{\partial u_1} \times \frac{\partial \mathsf{F}}{\partial u_2} = \det \frac{\partial(y,z)}{\partial(u_1,u_2)}\mathsf{i} + \det \frac{\partial(z,x)}{\partial(u_1,u_2)}\mathsf{j} + \det \frac{\partial(x,y)}{\partial(u_1,u_2)}\mathsf{k}$$

and hence the unit normal along this direction is given by:

$$\hat{\mathsf{n}}_\mathsf{F} = \frac{\det \frac{\partial(y,z)}{\partial(u_1,u_2)}\mathsf{i} + \det \frac{\partial(z,x)}{\partial(u_1,u_2)}\mathsf{j} + \det \frac{\partial(x,y)}{\partial(u_1,u_2)}\mathsf{k}}{\left|\det \frac{\partial(y,z)}{\partial(u_1,u_2)}\mathsf{i} + \det \frac{\partial(z,x)}{\partial(u_1,u_2)}\mathsf{j} + \det \frac{\partial(x,y)}{\partial(u_1,u_2)}\mathsf{k}\right|} \quad \text{on } \mathsf{F}(\mathcal{U}).$$

Note that the above $\hat{\mathsf{n}}$ is defined locally on the domain $\mathsf{F}(\mathcal{U})$.

Now given another local parametrization $(x, y, z) = \mathsf{G}(v_1, v_2) : \mathcal{V} \to M$, one can find a unit normal using G as well:

$$\hat{\mathsf{n}}_\mathsf{G} = \frac{\det \frac{\partial(y,z)}{\partial(v_1,v_2)}\mathsf{i} + \det \frac{\partial(z,x)}{\partial(v_1,v_2)}\mathsf{j} + \det \frac{\partial(x,y)}{\partial(v_1,v_2)}\mathsf{k}}{\left|\det \frac{\partial(y,z)}{\partial(v_1,v_2)}\mathsf{i} + \det \frac{\partial(z,x)}{\partial(v_1,v_2)}\mathsf{j} + \det \frac{\partial(x,y)}{\partial(v_1,v_2)}\mathsf{k}\right|} \quad \text{on } \mathsf{G}(\mathcal{V}).$$

Using the chain rule, we have the following relation between the Jacobian determinants:

$$\det \frac{\partial(*,**)}{\partial(v_1,v_2)} = \det \frac{\partial(u_1,u_2)}{\partial(v_1,v_2)} \det \frac{\partial(*,**)}{\partial(u_1,u_2)}$$

(here $*$ and $**$ mean any of the x, y and z) and therefore $\hat{\mathsf{n}}_\mathsf{F}$ and $\hat{\mathsf{n}}_\mathsf{G}$ are related by:

$$\hat{\mathsf{n}}_\mathsf{G} = \frac{\det \frac{\partial(u_1,u_2)}{\partial(v_1,v_2)}}{\left|\det \frac{\partial(u_1,u_2)}{\partial(v_1,v_2)}\right|}\hat{\mathsf{n}}_\mathsf{F}.$$

Therefore, if there is an overlap between local coordinates (u_1, u_2) and (v_1, v_2), the unit normal vectors $\hat{\mathsf{n}}_\mathsf{F}$ and $\hat{\mathsf{n}}_\mathsf{G}$ agree with each other on the overlap $\mathsf{F}(\mathcal{U}) \cap \mathsf{G}(\mathcal{V})$ if and only if $\det \dfrac{\partial(u_1,u_2)}{\partial(v_1,v_2)} > 0$ (equivalently, $\det D(\mathsf{F}^{-1} \circ \mathsf{G}) > 0$).

From above, we see that consistency of unit normal vector on different local coordinate charts is closely related to the positivity of the determinants of transition maps. A consistence choice of unit normal vector $\hat{\mathsf{n}}$ exists if and only if it is possible to pick a family of local parametrizations $\mathsf{F}_\alpha : \mathcal{U}_\alpha \to M^2$ covering the whole M such that $\det D(\mathsf{F}_\beta^{-1} \circ \mathsf{F}_\alpha) > 0$ on $\mathsf{F}_\alpha^{-1}\left(\mathsf{F}_\alpha(\mathcal{U}_\alpha) \cap \mathsf{F}_\beta(\mathcal{U}_\beta)\right)$ for any α and β in the family. The notion of normal vectors makes sense only for hypersurfaces in $\mathbb{R}^n$, while the notion of transition maps can extend to any abstract manifold.

Note that given two local parametrizations $\mathsf{F}(u_1, u_2)$ and $\mathsf{G}(v_1, v_2)$, it is *not* always possible to make sure $\det \dfrac{\partial(u_1, u_2)}{\partial(v_1, v_2)} > 0$ on the overlap even by switching v_1 and v_2. It is because it sometimes happens that the overlap $\mathsf{F}(\mathcal{U}) \cap \mathsf{G}(\mathcal{V})$ is a disjoint union of two open sets. If on one open set the determinant is positive, and on another one the determinant is negative, then switching v_1 and v_2 cannot make the determinant positive on both open sets. Let's illustrate this issue through two contrasting examples: the cylinder and the Möbius strip:

Example 4.10. The unit cylinder Σ^2 in $\mathbb{R}^3$ can be covered by two local parametrizations:

$$\begin{aligned} &\mathsf{F} : (0, 2\pi) \times \mathbb{R} \to \Sigma^2 & &\widetilde{\mathsf{F}} : (-\pi, \pi) \times \mathbb{R} \to \Sigma^2 \\ &\mathsf{F}(\theta, z) := (\cos\theta, \sin\theta, z) & &\widetilde{\mathsf{F}}(\widetilde{\theta}, \widetilde{z}) := (\cos\widetilde{\theta}, \sin\widetilde{\theta}, \widetilde{z}) \end{aligned}$$

Then, the transition map $\widetilde{\mathsf{F}}^{-1} \circ \mathsf{F}$ is defined on a disconnected domain $\theta \in (0, \pi) \cup (\pi, 2\pi)$ and $z \in \mathbb{R}$, and it is given by:

$$\widetilde{\mathsf{F}}^{-1} \circ \mathsf{F}(\theta, z) = \begin{cases} (\theta, z) & \text{if } \theta \in (0, \pi) \\ (\theta - 2\pi, z) & \text{if } \theta \in (\pi, 2\pi) \end{cases}$$

By direct computations, the Jacobian of this transition map is given by:

$$D(\widetilde{\mathsf{F}}^{-1} \circ \mathsf{F})(\theta, z) = I$$

in either case $\theta \in (0, \pi)$ or $\theta \in (\pi, 2\pi)$. Therefore, $\det D(\widetilde{\mathsf{F}}^{-1} \circ \mathsf{F}) > 0$ on the overlap.

The unit normal vectors defined using these F and $\widetilde{\mathsf{F}}$:

$$\begin{aligned} \hat{\mathsf{n}}_{\mathsf{F}} &= \frac{\frac{\partial \mathsf{F}}{\partial r} \times \frac{\partial \mathsf{F}}{\partial \theta}}{\left|\frac{\partial \mathsf{F}}{\partial r} \times \frac{\partial \mathsf{F}}{\partial \theta}\right|} \quad \text{on } \mathsf{F}((0, 2\pi) \times \mathbb{R}) \\ \hat{\mathsf{n}}_{\widetilde{\mathsf{F}}} &= \frac{\frac{\partial \widetilde{\mathsf{F}}}{\partial \widetilde{r}} \times \frac{\partial \widetilde{\mathsf{F}}}{\partial \widetilde{\theta}}}{\left|\frac{\partial \widetilde{\mathsf{F}}}{\partial \widetilde{r}} \times \frac{\partial \widetilde{\mathsf{F}}}{\partial \widetilde{\theta}}\right|} \quad \text{on } \widetilde{\mathsf{F}}((-\pi, \pi) \times \mathbb{R}) \end{aligned}$$

will agree with each other on the overlap. Therefore, it defines a *global* continuous unit normal vector across the whole cylinder. □

Example 4.11. The Möbius strip Σ^2 in $\mathbb{R}^3$ can be covered by two local parametrizations:

$$\begin{aligned} &\mathsf{F} : (-1, 1) \times (0, 2\pi) \to \Sigma^2 & &\widetilde{\mathsf{F}} : (-1, 1) \times (-\pi, \pi) \to \Sigma^2 \\ &\mathsf{F}(u, \theta) = \begin{bmatrix} \left(3 + u\cos\frac{\theta}{2}\right)\cos\theta \\ \left(3 + u\cos\frac{\theta}{2}\right)\sin\theta \\ u\sin\frac{\theta}{2} \end{bmatrix} & &\widetilde{\mathsf{F}}(\widetilde{u}, \widetilde{\theta}) = \begin{bmatrix} \left(3 + \widetilde{u}\cos\frac{\widetilde{\theta}}{2}\right)\cos\widetilde{\theta} \\ \left(3 + \widetilde{u}\cos\frac{\widetilde{\theta}}{2}\right)\sin\widetilde{\theta} \\ \widetilde{u}\sin\frac{\widetilde{\theta}}{2} \end{bmatrix} \end{aligned}$$

In order to compute the transition map $\widetilde{\mathsf{F}}^{-1} \circ \mathsf{F}(u, \theta)$, we need to solve the system of equations, i.e. find $(\widetilde{u}, \widetilde{\theta})$ in terms of (u, θ):

$$\left(3 + u\cos\frac{\theta}{2}\right)\cos\theta = \left(3 + \widetilde{u}\cos\frac{\widetilde{\theta}}{2}\right)\cos\widetilde{\theta} \tag{4.1}$$

$$\left(3 + u\cos\frac{\theta}{2}\right)\sin\theta = \left(3 + \widetilde{u}\cos\frac{\widetilde{\theta}}{2}\right)\sin\widetilde{\theta} \tag{4.2}$$

$$u\sin\frac{\theta}{2} = \widetilde{u}\sin\frac{\widetilde{\theta}}{2} \tag{4.3}$$

By considering $(4.1)^2 + (4.2)^2$, we get:

$$u\cos\frac{\theta}{2} = \widetilde{u}\cos\frac{\widetilde{\theta}}{2} \tag{4.4}$$

We leave it as an exercise for readers to check that $\theta \neq \pi$ in order for the system to be solvable. Therefore, $\theta \in (0, \pi) \cup (\pi, 2\pi)$ and so the domain of overlap is a disjoint union of two open sets.

When $\theta \in (0, \pi)$, from (4.3) and (4.4) we can conclude that $\widetilde{u} = u$ and $\widetilde{\theta} = \theta$.

When $\theta \in (\pi, 2\pi)$, we *cannot* have $\widetilde{\theta} = \theta$ since $\widetilde{\theta} \in (-\pi, \pi)$. However, one can have $\widetilde{u} = -u$ so that (4.3) and (4.4) become:

$$\sin\frac{\theta}{2} = -\sin\frac{\widetilde{\theta}}{2} \quad \text{and} \quad \cos\frac{\theta}{2} = -\cos\frac{\widetilde{\theta}}{2}$$

which implies $\widetilde{\theta} = \theta - 2\pi$.

To conclude, we have:

$$\widetilde{\mathsf{F}}^{-1} \circ \mathsf{F}(u, \theta) = \begin{cases} (u, \theta) & \text{if } \theta \in (0, \pi) \\ (-u, \theta - 2\pi) & \text{if } \theta \in (\pi, 2\pi) \end{cases}$$

By direct computations, we get:

$$\det D(\widetilde{\mathsf{F}}^{-1} \circ \mathsf{F})(u, \theta) = \begin{cases} 1 & \text{if } \theta \in (0, \pi) \\ -1 & \text{if } \theta \in (\pi, 2\pi) \end{cases}$$

Therefore, no matter how we switch the order of u and θ, or $\widetilde{u}$ and $\widetilde{\theta}$, we can never allow $\det D(\widetilde{\mathsf{F}}^{-1} \circ \mathsf{F}) > 0$ everywhere on the overlap. In other words, even if the unit normal vectors $\hat{\mathsf{n}}_{\mathsf{F}}$ and $\hat{\mathsf{n}}_{\widetilde{F}}$ agree with each other when $\theta \in (0, \pi)$, it would point in opposite direction when $\theta \in (\pi, 2\pi)$. □

Next, we are back to hypersurfaces M^n in $\mathbb{R}^{n+1}$ and prove the equivalence between consistency of unit normal and positivity of transition maps. To begin, we need the following result about normal vectors (which is left as an exercise for readers):

Exercise 4.6. Let M^n be a smooth hypersurface in $\mathbb{R}^{n+1}$ whose coordinates are denoted by $(x_1, \ldots, x_{n+1})$, and the unit vector along the x_i-direction is denoted by e_i. Let $\mathsf{F}(u_1, \ldots, u_n) : \mathcal{U} \to M^n$ be a local parametrization of M. Show that the following vector defined on $\mathsf{F}(\mathcal{U})$ is normal to the hypersurface M^n:

$$\sum_{i=1}^{n+1} \det \frac{\partial(x_{i+1}, \ldots, x_{n+1}, x_1, \ldots, x_{i-1})}{\partial(u_1, \ldots, u_n)} \mathsf{e}_i.$$

Proposition 4.12. *Given a smooth hypersurface M^n in $\mathbb{R}^{n+1}$, the following are equivalent:*

(i) *M^n is orientable;*

(ii) *There exists a family of local parametrizations $\mathsf{F}_\alpha : \mathcal{U}_\alpha \to M$ covering M such that for any $\mathsf{F}_\alpha, \mathsf{F}_\beta$ in the family with $\mathsf{F}_\beta(\mathcal{U}_\beta) \cap \mathsf{F}_\alpha(\mathcal{U}_\alpha) \neq \emptyset$, we have:*

$$\det D(\mathsf{F}_\alpha^{-1} \circ \mathsf{F}_\beta) > 0 \quad \text{on } \mathsf{F}_\beta^{-1}\left(\mathsf{F}_\beta(\mathcal{U}_\beta) \cap \mathsf{F}_\alpha(\mathcal{U}_\alpha)\right).$$

Proof. We first prove (ii) $\Longrightarrow$ (i). Denote $(u_1^\alpha, \ldots, u_n^\alpha)$ to be the local coordinates of M under the parametrization F_α. On every $\mathsf{F}_\alpha(\mathcal{U}_\alpha)$, using the result from Exercise 4.6, one can construct a unit normal vector locally defined on $\mathsf{F}_\alpha(\mathcal{U}_\alpha)$:

$$\hat{\mathsf{n}}_\alpha = \frac{\sum_{i=1}^{n+1} \det \frac{\partial(x_{i+1},\ldots,x_{n+1},x_1,\ldots,x_{i-1})}{\partial(u_1^\alpha,\ldots,u_n^\alpha)} \mathsf{e}_i}{\left|\sum_{i=1}^{n+1} \det \frac{\partial(x_{i+1},\ldots,x_{n+1},x_1,\ldots,x_{i-1})}{\partial(u_1^\alpha,\ldots,u_n^\alpha)} \mathsf{e}_i\right|}$$

Similarly, on $\mathsf{F}_\beta(\mathcal{U}_\beta)$, we have another locally defined unit normal vectors:

$$\hat{\mathsf{n}}_\beta = \frac{\sum_{i=1}^{n+1} \det \frac{\partial(x_{i+1},\ldots,x_{n+1},x_1,\ldots,x_{i-1})}{\partial(u_1^\beta,\ldots,u_n^\beta)} \mathsf{e}_i}{\left|\sum_{i=1}^{n+1} \det \frac{\partial(x_{i+1},\ldots,x_{n+1},x_1,\ldots,x_{i-1})}{\partial(u_1^\beta,\ldots,u_n^\beta)} \mathsf{e}_i\right|}$$

Then on the overlap $\mathsf{F}_\beta^{-1}\left(\mathsf{F}_\alpha(\mathcal{U}_\alpha) \cap \mathsf{F}_\beta(\mathcal{U}_\beta)\right)$, the chain rule asserts that:

$$\begin{aligned}
&\det \frac{\partial(x_{i+1},\ldots,x_{n+1},x_1,\ldots,x_{i-1})}{\partial(u_1^\beta,\ldots,u_n^\beta)} \\
&= \det \frac{\partial(u_1^\alpha,\ldots,u_n^\alpha)}{\partial(u_1^\beta,\ldots,u_n^\beta)} \det \frac{\partial(x_{i+1},\ldots,x_{n+1},x_1,\ldots,x_{i-1})}{\partial(u_1^\alpha,\ldots,u_n^\alpha)} \\
&= \det D(\mathsf{F}_\alpha^{-1} \circ \mathsf{F}_\beta) \det \frac{\partial(x_{i+1},\ldots,x_{n+1},x_1,\ldots,x_{i-1})}{\partial(u_1^\alpha,\ldots,u_n^\alpha)}
\end{aligned}$$

and so the two unit normal vectors are related by:

$$\hat{\mathsf{n}}_\beta = \frac{\det D(\mathsf{F}_\alpha^{-1} \circ \mathsf{F}_\beta)}{\left|\det D(\mathsf{F}_\alpha^{-1} \circ \mathsf{F}_\beta)\right|} \hat{\mathsf{n}}_\alpha.$$

By the condition that $\det D(\mathsf{F}_\alpha^{-1} \circ \mathsf{F}_\beta) > 0$, we have $\hat{\mathsf{n}}_\beta = \hat{\mathsf{n}}_\alpha$ on the overlap. Define $\hat{\mathsf{n}} := \hat{\mathsf{n}}_\alpha$ on every $\mathsf{F}_\alpha(\mathcal{U}_\alpha)$, it is then a continuous unit normal vector globally defined on M. This proves (i).

Now we show (i) $\Longrightarrow$ (ii). Suppose $\hat{\mathsf{n}}$ is a continuous unit normal vector defined on the whole M. Suppose $\mathsf{F}_\alpha(u_1^\alpha, \ldots, u_n^\alpha) : \mathcal{U}_\alpha \to M$ is *any* family of local parametrizations that cover the whole M. On every $\mathsf{F}_\alpha(\mathcal{U}_\alpha)$, we consider the locally defined unit normal vector:

$$\hat{\mathsf{n}}_\alpha = \frac{\sum_{i=1}^{n+1} \det \frac{\partial(x_{i+1},\ldots,x_{n+1},x_1,\ldots,x_{i-1})}{\partial(u_1^\alpha,\ldots,u_n^\alpha)} \mathsf{e}_i}{\left|\sum_{i=1}^{n+1} \det \frac{\partial(x_{i+1},\ldots,x_{n+1},x_1,\ldots,x_{i-1})}{\partial(u_1^\alpha,\ldots,u_n^\alpha)} \mathsf{e}_i\right|}.$$

As a hypersurface M^n in $\mathbb{R}^{n+1}$, there is only one direction of normal vectors, and so we have either $\hat{\mathsf{n}}_\alpha = \hat{\mathsf{n}}$ or $\hat{\mathsf{n}}_\alpha = -\hat{\mathsf{n}}$ on $\mathsf{F}_\alpha(\mathcal{U}_\alpha)$. For the latter case, one can modify the parametrization F_α by switching any pair of u_i^α's such that $\hat{\mathsf{n}}_\alpha = \hat{\mathsf{n}}$.

After making suitable modification on every F_α, we can assume without loss of generality that F_α's are local parametrizations such that $\hat{\mathsf{n}}_\alpha = \hat{\mathsf{n}}$ on every $\mathsf{F}_\alpha(\mathcal{U}_\alpha)$. In particular, on the overlap $\mathsf{F}_\beta^{-1}\left(\mathsf{F}_\alpha(\mathcal{U}_\alpha) \cap \mathsf{F}_\beta(\mathcal{U}_\beta)\right)$, we have $\hat{\mathsf{n}}_\alpha = \hat{\mathsf{n}}_\beta$.

By $\hat{\mathsf{n}}_\beta = \dfrac{\det D(\mathsf{F}_\alpha^{-1} \circ \mathsf{F}_\beta)}{\left|\det D(\mathsf{F}_\alpha^{-1} \circ \mathsf{F}_\beta)\right|}\hat{\mathsf{n}}_\alpha$, we conclude that $\det D(\mathsf{F}_\alpha^{-1} \circ \mathsf{F}_\beta) > 0$, proving (ii).

□

Remark 4.13. According to Proposition 4.12, the cylinder in Example 4.10 is orientable, while the Möbius strip in Example 4.11 is not orientable. □

Exercise 4.7. Show that the unit sphere $\mathbb{S}^2$ in $\mathbb{R}^3$ is orientable.

Exercise 4.8. Let $f : \mathbb{R}^3 \to \mathbb{R}$ be a smooth function. Suppose $c \in \mathbb{R}$ such that $f^{-1}(c)$ is non-empty and f is a submersion at every $p \in f^{-1}(c)$. Show that $f^{-1}(c)$ is an orientable hypersurface in $\mathbb{R}^3$.

4.2.2. Orientable Manifolds. On an abstract manifold M, it is not possible to define normal vectors on M, and so the notion of orientability cannot be defined using normal vectors. However, thanks to Proposition 4.12, the notion of orientability of hypersurfaces is equivalent to positivity of Jacobians of transition maps, which we can also talk about on abstract manifolds. Therefore, motivated by Proposition 4.12, we define:

Definition 4.14 (Orientable Manifolds)**.** A smooth manifold M is said to be *orientable* if there exists a family of local parametrizations $\mathsf{F}_\alpha : \mathcal{U}_\alpha \to M$ covering M such that for any F_α and F_β in the family with $\mathsf{F}_\beta(\mathcal{U}_\beta) \cap \mathsf{F}_\alpha(\mathcal{U}_\alpha) \neq \emptyset$, we have:

$$\det D(\mathsf{F}_\alpha^{-1} \circ \mathsf{F}_\beta) > 0 \quad \text{on } \mathsf{F}_\beta^{-1}\left(\mathsf{F}_\beta(\mathcal{U}_\beta) \cap \mathsf{F}_\alpha(\mathcal{U}_\alpha)\right).$$

In this case, we call the family $\mathcal{A} = \{\mathsf{F}_\alpha : \mathcal{U}_\alpha \to M\}$ of local parametrizations to be an *oriented atlas* of M.

Example 4.15. Recall that the real projective space $\mathbb{RP}^2$ consists of homogeneous triples $[x_0 : x_1 : x_2]$ where $(x_0, x_1, x_2) \neq (0,0,0)$. The standard parametrizations are given by:

$$\mathsf{F}_0(x_1, x_2) = [1 : x_1 : x_2]$$
$$\mathsf{F}_1(y_0, y_2) = [y_0 : 1 : y_2]$$
$$\mathsf{F}_2(z_0, z_1) = [z_0 : z_1 : 1]$$

By the fact that $[y_0 : 1 : y_2] = [1 : y_0^{-1} : y_2 y_0^{-1}]$, the transition map $\mathsf{F}_0^{-1} \circ \mathsf{F}_1$ is defined on $\{(y_0, y_2) \in \mathbb{R}^2 : y_0 \neq 0\}$, and is given by: $(x_1, x_2) = (y_0^{-1}, y_2 y_0^{-1})$. Hence,

$$D(\mathsf{F}_0^{-1} \circ \mathsf{F}_1) = \frac{\partial(x_1, x_2)}{\partial(y_0, y_2)} = \begin{bmatrix} -y_0^{-2} & 0 \\ -y_2 y_0^{-2} & y_0^{-1} \end{bmatrix}$$
$$\det D(\mathsf{F}_0^{-1} \circ \mathsf{F}_1) = -\frac{1}{y_0^3}$$

Therefore, it is impossible for $\det D(\mathsf{F}_0^{-1} \circ \mathsf{F}_1) > 0$ on the overlap domain $\{(y_0, y_2) \in \mathbb{R}^2 : y_0 \neq 0\}$. We conclude that $\mathbb{RP}^2$ is not orientable. □

Exercise 4.9. Show that $\mathbb{RP}^3$ is orientable. Propose a conjecture about the orientability of $\mathbb{RP}^n$.

Exercise 4.10. Show that for *any* smooth manifold M (whether or not it is orientable), the tangent bundle TM must be orientable.

Exercise 4.11. Show that for a smooth orientable manifold M with boundary, the boundary manifold ∂M must also be orientable.

4.3. Integrations of Differential Forms

Generalized Stokes' Theorem concerns about integrals of differential forms. In this section, we will give a rigorous definition of these integrals.

4.3.1. Single Parametrization. In the *simplest* case if a manifold M can be covered by a single parametrization:

$$\mathsf{F}(u_1, \ldots, u_n) : (\alpha_1, \beta_1) \times \cdots \times (\alpha_n, \beta_n) \to M,$$

then given an n-form $\varphi(u_1, \ldots, u_n)\, du^1 \wedge du^2 \wedge \cdots \wedge du^n$, the integral of ω over the manifold M is given by:

$$\underbrace{\int_M \varphi(u_1, \ldots, u_n)\, du^1 \wedge du^2 \wedge \cdots \wedge du^n}_{\text{integral of differential form}} := \underbrace{\int_{\alpha_n}^{\beta_n} \cdots \int_{\alpha_1}^{\beta_1} \varphi(u_1, \ldots, u_n)\, du^1\, du^2 \cdots du^n}_{\text{ordinary integral in Multivariable Calculus}}$$

From the definition, we see that it only makes sense to integrate an n-form on an n-dimensional manifold.

Very few manifolds can be covered by a single parametrization. Of course, $\mathbb{R}^n$ is an example. One less trivial example is the graph of a smooth function. Suppose $f(x, y) : \mathbb{R}^2 \to \mathbb{R}$ is a smooth function. Consider its graph:

$$\Gamma_f := \{(x, y, f(x, y)) \in \mathbb{R}^3 : (x, y) \in \mathbb{R}^2\}$$

which can be globally parametrized by $\mathsf{F} : \mathbb{R}^2 \to \Gamma_f$ where

$$\mathsf{F}(x, y) = (x, y, f(x, y)).$$

Let $\omega = e^{-x^2-y^2}\, dx \wedge dy$ be a 2-form on Γ_f, then its integral over Γ_f is given by:

$$\int_{\Gamma_f} \omega = \int_{\Gamma_f} e^{-x^2-y^2}\, dx \wedge dy = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-x^2-y^2}\, dx\, dy = \pi.$$

Here we leave the computational detail as an exercise for readers.

It *appears* that integrating a differential form is just like "erasing the wedges", yet there are two subtle (but important) issues:

(1) In the above example, note that ω can also be written as:

$$\omega = -e^{-x^2-y^2}\, dy \wedge dx.$$

It suggests that we also have:

$$\int_{\Gamma_f} \omega = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} -e^{-x^2-y^2}\, dy\, dx = -\pi,$$

which is *not* consistent with the previous result. How shall we fix it?

(2) Even if a manifold can be covered by one single parametrization, such a parametrization may not be unique. If both $(u_1, \ldots, u_n)$ and $(v_1, \ldots, v_n)$ are global coordinates of M, then a differential form ω can be expressed in terms of either u_i's or v_i's. Is the integral independent of the chosen coordinate system?

The first issue can be resolved easily. Whenever we talk about integration of differential forms, we need to first fix the order of the coordinates. Say on $\mathbb{R}^2$ we fix the order to be (x, y), then for any given 2-form we should express it in terms of $dx \wedge dy$ before "erasing the wedges". For the 2-form ω above, we must first express it as:

$$\omega = e^{-x^2-y^2}\, dx \wedge dy$$

before integrating it.

For higher (say dim $= 4$) dimensional manifolds M^4 covered by a single parametrization $\mathsf{F}(u_1,\ldots,u_4) : \mathcal{U} \to M$, if we choose (u_1,u_2,u_3,u_4) to be the order of coordinates and given a 4-form:

$$\Omega = f(u_1,\ldots,u_4)\, du^1 \wedge du^3 \wedge du^2 \wedge du^4 + g(u_1,\ldots,u_4)\, du^4 \wedge du^3 \wedge du^2 \wedge du^1,$$

then we need to re-order the wedge product so that:

$$\Omega = -f(u_1,\ldots,u_4)\, du^1 \wedge du^2 \wedge du^3 \wedge du^4 + g(u_1,\ldots,u_4) du^1 \wedge du^2 \wedge du^3 \wedge du^4.$$

The integral of ω over M^4 with respect to the order (u_1,u_2,u_3,u_4) is given by:

$$\int_M \Omega = \int_{\mathcal{U}} \left(-f(u_1,\ldots,u_4) + g(u_1,\ldots,u_4)\right)\, du^1\, du^2\, du^3\, du^4.$$

Let's examine the second issue. Suppose M is an n-manifold with two different global parametrizations $\mathsf{F}(u_1,\ldots,u_n) : \mathcal{U} \to M$ and $\mathsf{G}(v_1,\ldots,v_n) : \mathcal{V} \to M$. Given an n-form ω which can be expressed as:

$$\omega = \varphi\, du^1 \wedge \cdots \wedge du^n,$$

then from Proposition 3.49, ω can be expressed in terms of v_i's by:

$$\omega = \varphi \, \det \frac{\partial(u_1,\ldots,u_n)}{\partial(v_1,\ldots,v_n)}\, dv^1 \wedge \cdots \wedge dv^n.$$

Recall that the change-of-variable formula in Multivariable Calculus asserts that:

$$\int_{\mathcal{U}} \varphi\, du^1 \cdots du^n = \int_{\mathcal{V}} \varphi \left| \det \frac{\partial(u_1,\ldots,u_n)}{\partial(v_1,\ldots,v_n)} \right| dv^1 \cdots dv^n.$$

Therefore, in order for $\int_M \omega$ to be well-defined, we need

$$\int_{\mathcal{U}} \varphi\, du^1 \wedge \cdots \wedge du^n \text{ and } \int_{\mathcal{V}} \varphi \, \det \frac{\partial(u_1,\ldots,u_n)}{\partial(v_1,\ldots,v_n)}\, dv^1 \wedge \cdots \wedge dv^n$$

to be equal, and so we require:

$$\det \frac{\partial(u_1,\ldots,u_n)}{\partial(v_1,\ldots,v_n)} > 0.$$

When defining an integral of a differential form, we *not only* need to choose a convention on the order of coordinates, say $(u_1,\ldots,u_n)$, but also we shall only consider those coordinate systems $(v_1,\ldots,v_n)$ such that $\det \frac{\partial(u_1,\ldots,u_n)}{\partial(v_1,\ldots,v_n)} > 0$. Therefore, in order to integrate a differential form, we require the manifold to be *orientable*.

4.3.2. Multiple Parametrizations. A majority of smooth manifolds are covered by more than one parametrizations. Integrating a differential form over such a manifold is not as straight-forward as previously discussed.

In case M can be "almost" covered by a single parametrization $\mathsf{F} : \mathcal{U} \to M$ (i.e. the set $M \backslash \mathsf{F}(\mathcal{U})$ has measure zero) and the n-form ω is continuous, then it is still possible to compute $\int_M \omega$ by computing $\int_{\mathsf{F}(\mathcal{U})} \omega$. Let's consider the example of a sphere:

Example 4.16. Let $\mathbb{S}^2$ be the unit sphere in $\mathbb{R}^3$ centered at the origin. Consider the 2-form ω on $\mathbb{R}^3$ defined as:

$$\omega = dx \wedge dy.$$

Let $\iota : \mathbb{S}^2 \to \mathbb{R}^3$ be the inclusion map, then $\iota^*\omega$ is a 2-form on $\mathbb{S}^2$. We are interested in the value of the integral $\int_{\mathbb{S}^2} \iota^*\omega$.

Note that $\mathbb{S}^2$ can be covered almost everywhere by spherical coordinate parametrization $\mathsf{F}(\varphi, \theta) : (0, \pi) \times (0, 2\pi) \to \mathbb{S}^2$ given by:

$$\mathsf{F}(\varphi, \theta) = (\sin\varphi\cos\theta, \sin\varphi\sin\theta, \cos\varphi).$$

Under the local coordinates (φ, θ), we have:

$$\begin{aligned}
\iota^*(dx) &= d(\sin\varphi\cos\theta) = \cos\varphi\cos\theta\, d\varphi - \sin\varphi\sin\theta\, d\theta \\
\iota^*(dy) &= d(\sin\varphi\sin\theta) = \cos\varphi\sin\theta\, d\varphi + \sin\varphi\cos\theta\, d\theta \\
\iota^*\omega &= \iota^*(dx) \wedge \iota^*(dy) \\
&= \sin\varphi\cos\varphi\, d\varphi \wedge d\theta.
\end{aligned}$$

Therefore,

$$\int_M \iota^*\omega = \int_M \sin\varphi\cos\varphi\, d\varphi \wedge d\theta = \int_0^{2\pi}\int_0^{\pi} \sin\varphi\cos\varphi\, d\varphi\, d\theta = 0.$$

Here we pick (φ, θ) as the order of coordinates. □

Exercise 4.12. Let $\omega = x\, dy \wedge dz + y\, dz \wedge dx + z\, dx \wedge dy$. Compute

$$\int_{\mathbb{S}^2} \iota^*\omega$$

where $\mathbb{S}^2$ is the unit sphere in $\mathbb{R}^3$ centered at the origin, and $\iota : \mathbb{S}^2 \to \mathbb{R}^3$ is the inclusion map.

Exercise 4.13. Let $\mathbb{T}^2$ be the torus in $\mathbb{R}^4$ defined as:

$$\mathbb{T}^2 := \left\{ (x_1, x_2, x_3, x_4) \in \mathbb{R}^4 : x_1^2 + x_2^2 = x_3^2 + x_4^2 = \frac{1}{2} \right\}.$$

Let $\iota : \mathbb{T}^2 \to \mathbb{R}^4$ be the inclusion map. Compute the following integral:

$$\int_{\mathbb{T}^2} \iota^* \left(x_1 x_2 x_3\, dx^4 \wedge dx^3 \right).$$

FYI: Clifford Torus

The torus $\mathbb{T}^2$ in Exercise 4.13 is a well-known object in Differential Geometry called the *Clifford Torus*. A famous conjecture called the Hsiang-Lawson's Conjecture concerns about this torus. One of the proposers Wu-Yi Hsiang is a retired faculty of HKUST Math. This conjecture was recently solved by Simon Brendle in 2012.

Next, we will discuss how to define integrals of differential forms when M is covered by multiple parametrizations none of which can almost cover the whole manifold. The key idea is to break down the n-form into small pieces, so that each piece is completely covered by one single parametrization. It will be done using *partition of unity* to be discussed.

We first introduce the notion of *support* which appears often in the rest of the course (as well as in advanced PDE courses).

Definition 4.17 (Support)**.** Let M be a smooth manifold. Given a k-form ω (where $0 \le k \le n$) defined on M, we denote and define the *support of* ω to be:

$$\operatorname{supp}\omega := \overline{\{p \in M : \omega(p) \neq 0\}},$$

i.e. the *closure* of the set $\{p \in M : \omega(p) \neq 0\}$.

Suppose M^n is an oriented manifold with $\mathsf{F}(u_1, \ldots, u_n) : \mathcal{U} \to M$ as one of (many) local parametrizations. If an n-form ω on M^n only has "stuff" inside $\mathsf{F}(\mathcal{U})$, or precisely:

$$\operatorname{supp} \omega \subset \mathsf{F}(\mathcal{U}),$$

then one can define $\int_M \omega$ as in the previous subsection. Namely, if on $\mathsf{F}(\mathcal{U})$ we have $\omega = \varphi \, du^1 \wedge \cdots \wedge du^n$, then we define:

$$\int_M \omega = \int_{\mathsf{F}(\mathcal{U})} \omega = \int_{\mathcal{U}} \varphi \, du^1 \cdots du^n.$$

Here we pick the order of coordinates to be $(u_1, \ldots, u_n)$.

The following important tool called *partitions of unity* will "chop" a differential form into "little pieces" such that each piece has support covered by a single parametrization.

Definition 4.18 (Partitions of Unity). Let M be a smooth manifold with an atlas $\mathcal{A} = \{\mathsf{F}_\alpha : \mathcal{U}_\alpha \to M\}$ such that $M = \bigcup_{\text{all } \alpha} \mathsf{F}_\alpha(\mathcal{U}_\alpha)$. A *partition of unity subordinate to the atlas* $\mathcal{A}$ is a family of smooth functions $\rho_\alpha : M \to [0,1]$ with the following properties:

(i) $\operatorname{supp} \rho_\alpha \subset \mathsf{F}_\alpha(\mathcal{U}_\alpha)$ for any α.

(ii) For any $p \in M$, there exists an open set $\mathcal{O} \subset M$ containing p such that

$$\operatorname{supp} \rho_\alpha \cap \mathcal{O} \neq \varnothing$$

for finitely many α's only.

(iii) $\sum_{\text{all } \alpha} \rho_\alpha \equiv 1$ on M.

Remark 4.19. It can be shown that given any smooth manifold with any atlas, partitions of unity subordinate to that given atlas must exist. The proof is very technical and is not in the same spirit with other parts of the course, so we omit the proof here. It is more important to know what partitions of unity are for, than to know the proof of existence. □

Remark 4.20. Note that partitions of unity subordinate to a given atlas may not be unique! □

Remark 4.21. Condition (ii) in Definition 4.18 is merely a technical analytic condition to make sure the sum $\sum_{\text{all } \alpha} \rho_\alpha(p)$ is a finite sum for each fixed $p \in M$, so that we do not need to worry about convergence issues. If the manifold can be covered by finitely many local parametrizations, then condition (ii) automatically holds (and we do not need to worry about). □

Now, take an n-form ω defined on an orientable manifold M^n, which is parametrized by an oriented atlas $\mathcal{A} = \{\mathsf{F}_\alpha : \mathcal{U}_\alpha \to M\}$. Let $\{\rho_\alpha : M \to [0,1]\}$ be a partition of unity subordinate to $\mathcal{A}$, then by condition (iii) in Definition 4.18, we get:

$$\omega = \underbrace{\left(\sum_{\text{all } \alpha} \rho_\alpha \right)}_{=1} \omega = \sum_{\text{all } \alpha} \rho_\alpha \omega.$$

Condition (i) says that $\operatorname{supp} \rho_\alpha \subset \mathsf{F}_\alpha(\mathcal{U}_\alpha)$, or heuristically speaking ρ_α vanishes outside $\mathsf{F}_\alpha(\mathcal{U}_\alpha)$. Naturally, we have $\operatorname{supp}(\rho_\alpha \omega) \subset \mathsf{F}_\alpha(\mathcal{U}_\alpha)$ for each α. Therefore, as previously discussed, we can integrate $\rho_\alpha \omega$ for each *individual* α:

$$\int_M \rho_\alpha \omega := \int_{\mathsf{F}_\alpha(\mathcal{U}_\alpha)} \rho_\alpha \omega.$$

Given that we can integrate each $\rho_\alpha\omega$, we define the integral of ω as:

$$\int_M \omega := \sum_{\text{all } \alpha} \int_M \rho_\alpha \omega = \sum_{\text{all } \alpha} \int_{F_\alpha(\mathcal{U}_\alpha)} \rho_\alpha \omega. \tag{4.5}$$

However, the sum involved in (4.5) is in general an infinite (possible uncountable!) sum. To avoid convergence issue, from now on we will only consider n-forms ω which have *compact support*, i.e.

$$\operatorname{supp}\omega \text{ is a compact set.}$$

Recall that every open cover of a compact set has a finite sub-cover. Together with condition (ii) in Definition 4.18, one can show that $\rho_\alpha\omega$ are identically zero for all except finitely many α's. The argument goes as follows: at each $p \in \operatorname{supp}\omega$, by condition (ii) in Definition 4.18, there exists an open set $\mathcal{O}_p \subset M$ containing p such that the set:

$$S_p := \{\alpha : \operatorname{supp}\rho_\alpha \cap \mathcal{O}_p \neq \emptyset\}$$

is finite. Evidently, we have

$$\operatorname{supp}\omega \subset \bigcup_{p\in\operatorname{supp}\omega} \mathcal{O}_p$$

and by compactness of $\operatorname{supp}\omega$, there exists $p_1, \ldots, p_N \in \operatorname{supp}\omega$ such that

$$\operatorname{supp}\omega \subset \bigcup_{i=1}^N \mathcal{O}_{p_i}.$$

Since $\{q \in M : \rho_\alpha(q)\omega(q) \neq 0\} \subset \{q \in M : \rho_\alpha(q) \neq 0\} \cap \{q \in M : \omega(q) \neq 0\}$, we have:

$$\begin{aligned}
\operatorname{supp}(\rho_\alpha\omega) &= \overline{\{q \in M : \rho_\alpha(q)\omega(q) \neq 0\}} \\
&\subset \overline{\{q \in M : \rho_\alpha(q) \neq 0\} \cap \{q \in M : \omega(q) \neq 0\}} \\
&\subset \overline{\{q \in M : \rho_\alpha(q) \neq 0\}} \cap \overline{\{q \in M : \omega(q) \neq 0\}} \\
&= \operatorname{supp}\rho_\alpha \cap \operatorname{supp}\omega \subset \bigcup_{i=1}^N \left(\operatorname{supp}\rho_\alpha \cap \mathcal{O}_{p_i}\right).
\end{aligned}$$

Therefore, if α is an index such that $\operatorname{supp}(\rho_\alpha\omega) \neq \emptyset$, then there exists $i \in \{1, \ldots, N\}$ such that $\operatorname{supp}\rho_\alpha \cap \mathcal{O}_{p_i} \neq \emptyset$, or in other words, $\alpha \in S_{p_i}$ for some i, and so:

$$\{\alpha : \operatorname{supp}(\rho_\alpha\omega) \neq \emptyset\} \subset \bigcup_{i=1}^N S_{p_i}.$$

Since each S_{p_i} is a finite set, the set $\{\alpha : \operatorname{supp}(\rho_\alpha\omega) \neq \emptyset\}$ is also finite. Therefore, there are only finitely many α's such that $\int_{F_\alpha(\mathcal{U}_\alpha)}$ is non-zero, and so the sum stated in (4.5) is in fact a *finite sum*.

Now we have understood that there is no convergence issue for (4.5) provided that ω has compact support (which is automatically true if the manifold M is itself compact). There are still two well-definedness issues to resolve, namely whether the integral in (4.5) is independent of oriented atlas $\mathcal{A}$, and for each atlas whether the integral is independent of the choice of partitions of unity.

Proposition 4.22. *Let M^n be an orientable smooth manifold with two oriented atlas*

$$\mathcal{A} = \{\mathsf{F}_\alpha : \mathcal{U}_\alpha \to M\} \text{ and } \mathcal{B} = \{\mathsf{G}_\beta : \mathcal{V}_\beta \to M\}$$

such that $\det D(\mathsf{F}_\alpha^{-1} \circ \mathsf{G}_\beta) > 0$ *on the overlap for any pair of α and β. Suppose $\{\rho_\alpha : M \to [0,1]\}$ and $\{\sigma_\beta : M \to [0,1]\}$ are partitions of unity subordinate to $\mathcal{A}$ and $\mathcal{B}$ respectively. Then, given any compactly supported differential n-form ω on M^n, we have:*

$$\sum_{\text{all } \alpha} \int_{\mathsf{F}_\alpha(\mathcal{U}_\alpha)} \rho_\alpha \omega = \sum_{\text{all } \beta} \int_{\mathsf{G}_\beta(\mathcal{V}_\beta)} \sigma_\beta \omega.$$

Proof. By the fact that $\sum_{\text{all } \beta} \sigma_\beta \equiv 1$ on M, we have:

$$\sum_{\text{all } \alpha} \int_{\mathsf{F}_\alpha(\mathcal{U}_\alpha)} \rho_\alpha \omega = \sum_{\text{all } \alpha} \int_{\mathsf{F}_\alpha(\mathcal{U}_\alpha)} \left(\sum_{\text{all } \beta} \sigma_\beta \right) \rho_\alpha \omega = \sum_{\text{all } \alpha} \sum_{\text{all } \beta} \int_{\mathsf{F}_\alpha(\mathcal{U}_\alpha) \cap \mathsf{G}_\beta(\mathcal{V}_\beta)} \rho_\alpha \sigma_\beta \omega.$$

The last equality follows from the fact that $\operatorname{supp} \sigma_\beta \subset \mathsf{G}_\beta(\mathcal{V}_\beta)$.

One can similarly work out that

$$\sum_{\text{all } \beta} \int_{\mathsf{G}_\beta(\mathcal{V}_\beta)} \sigma_\beta \omega = \sum_{\text{all } \beta} \sum_{\text{all } \alpha} \int_{\mathsf{F}_\alpha(\mathcal{U}_\alpha) \cap \mathsf{G}_\beta(\mathcal{V}_\beta)} \rho_\alpha \sigma_\beta \omega.$$

Note that $\sum_\alpha \sum_\beta$ is a finite double sum and so there is no issue of switching them. It completes the proof. □

By Proposition 4.22, we justified that (4.5) is independent of oriented atlas and the choice of partitions of unity. We can now define:

Definition 4.23. Let M^n be an orientable smooth manifold with an oriented atlas $\mathcal{A} = \{\mathsf{F}_\alpha(u_\alpha^1, \ldots, u_\alpha^n) : \mathcal{U}_\alpha \to M\}$ where $(u_\alpha^1, \ldots, u_\alpha^n)$ is the chosen order of local coordinates. Pick a partition of unity $\{\rho_\alpha : M \to [0,1]\}$ subordinate to the atlas $\mathcal{A}$. Then, given any n-form ω, we define its *integral over M* as:

$$\int_M \omega := \sum_{\text{all } \alpha} \int_{\mathsf{F}_\alpha(\mathcal{U}_\alpha)} \rho_\alpha \omega.$$

If $\omega = \varphi_\alpha \, du_\alpha^1 \wedge \cdots \wedge du_\alpha^n$ on each $\mathsf{F}_\alpha(\mathcal{U}_\alpha)$, then:

$$\int_M \omega = \sum_{\text{all } \alpha} \int_{\mathcal{U}_\alpha} \rho_\alpha \varphi_\alpha \, du_\alpha^1 \cdots du_\alpha^n.$$

Remark 4.24. It is generally impossible to compute such an integral, as we know only the existence of ρ_α's but typically not the exact expressions. Even if such a partition of unity ρ_α's can be found, it often involves some terms such as e^{-1/x^2}, which is almost impossible to integrate. To conclude, we do not attempt compute such an integral, but we will study the properties of it based on the definition. □

4.3.3. Orientation of Manifolds. Partition of unity is a powerful tool to construct a smooth *global* item from *local* ones. For integrals of differential forms, we first defines integral of forms with support contained in a single parametrization chart, then we uses a partition of unity to *glue* each chart together. There are some other uses in this spirit. The following beautiful statement can be proved using partitions of unity:

Proposition 4.25. *A smooth n-dimensional manifold M is orientable if and only if there exists a non-vanishing smooth n-form globally defined on M.*

Proof. Suppose M is orientable, then by definition there exists an oriented atlas $\mathcal{A} = \{\mathsf{F}_\alpha : \mathcal{U}_\alpha \to M\}$ such that $\det D(\mathsf{F}_\beta^{-1} \circ \mathsf{F}_\alpha) > 0$ for any α and β. For each local parametrization F_α, we denote $(u_\alpha^1, \ldots, u_\alpha^n)$ to be its local coordinates, then the n-form:

$$\eta_\alpha := du_\alpha^1 \wedge \cdots \wedge du_\alpha^n$$

is *locally defined* on $\mathsf{F}_\alpha(\mathcal{U}_\alpha)$.

Let $\{\rho_\alpha : M \to [0,1]\}$ be a partition of unity subordinate to $\mathcal{A}$. We define:

$$\omega = \sum_{\text{all } \alpha} \rho_\alpha \eta_\alpha = \sum_{\text{all } \alpha} \rho_\alpha \, du_\alpha^1 \wedge \cdots \wedge du_\alpha^n.$$

We claim $\omega(p) \neq 0$ at every point $p \in M$. Suppose $p \in \mathsf{F}_\beta(\mathcal{U}_\beta)$ for some β in the atlas. By (3.12), for each α, locally near p we have:

$$du_\alpha^1 \wedge \cdots \wedge du_\alpha^n = \det \frac{\partial(u_\alpha^1, \ldots, u_\alpha^n)}{\partial(u_\beta^1, \ldots, u_\beta^n)} \, du_\beta^1 \wedge \cdots \wedge du_\beta^n,$$

and so:

$$\omega = \left(\sum_{\text{all } \alpha} \rho_\alpha \, \det \frac{\partial(u_\alpha^1, \ldots, u_\alpha^n)}{\partial(u_\beta^1, \ldots, u_\beta^n)} \right) du_\beta^1 \wedge \cdots \wedge du_\beta^n.$$

Since $\rho_\alpha \geq 0$, $\sum_{\text{all } \alpha} \rho_\alpha \equiv 1$ and $\det \frac{\partial(u_\alpha^1, \ldots, u_\alpha^n)}{\partial(u_\beta^1, \ldots, u_\beta^n)} > 0$, we must have:

$$\sum_{\text{all } \alpha} \rho_\alpha \, \det \frac{\partial(u_\alpha^1, \ldots, u_\alpha^n)}{\partial(u_\beta^1, \ldots, u_\beta^n)} > 0 \quad \text{near } p.$$

This shows ω is a non-vanishing n-form on M.

Conversely, suppose Ω is a non-vanishing n-form on M. Let $\mathcal{C} = \{\mathsf{G}_\alpha : \mathcal{V}_\alpha \to M\}$ be any atlas on M, and for each α we denote $(v_\alpha^1, \ldots, v_\alpha^n)$ to be its local coordinates. Express Ω in terms of local coordinates:

$$\Omega = \varphi_\alpha \, dv_\alpha^1 \wedge \cdots \wedge dv_\alpha^n.$$

Since Ω is non-vanishing, φ_α must be either positive on $\mathcal{V}_\alpha$, or negative on $\mathcal{V}_\alpha$. Re-define the local coordinates by:

$$(\widetilde{v}_\alpha^1, \widetilde{v}_\alpha^2, \ldots, \widetilde{v}_\alpha^n) := \begin{cases} (v_\alpha^1, v_\alpha^2, \ldots, v_\alpha^n) & \text{if } \varphi_\alpha > 0 \\ (-v_\alpha^1, v_\alpha^2, \ldots, v_\alpha^n) & \text{if } \varphi_\alpha < 0 \end{cases}$$

Then, under these new local coordinates, we have:

$$\Omega = |\varphi_\alpha| \, d\widetilde{v}_\alpha^1 \wedge \cdots \wedge d\widetilde{v}_\alpha^n.$$

From (3.12), we can deduce:

$$\Omega = |\varphi_\alpha| \, d\widetilde{v}_\alpha^1 \wedge \cdots \wedge d\widetilde{v}_\alpha^n = |\varphi_\alpha| \, \det \frac{\partial(\widetilde{v}_\alpha^1, \ldots, \widetilde{v}_\alpha^n)}{\partial(\widetilde{v}_\beta^1, \ldots, \widetilde{v}_\beta^n)} \, d\widetilde{v}_\beta^1 \wedge \cdots \wedge d\widetilde{v}_\beta^n$$

on the overlap of any two local coordinates $(\widetilde{v}_\alpha^1, \ldots, \widetilde{v}_\alpha^n)$ and $(\widetilde{v}_\beta^1, \ldots, \widetilde{v}_\beta^n)$. On the other hand, we have:

$$\Omega = \left|\varphi_\beta\right| \, d\widetilde{v}_\beta^1 \wedge \cdots \wedge d\widetilde{v}_\beta^n.$$

This shows:

$$\det \frac{\partial(\widetilde{v}_\alpha^1, \ldots, \widetilde{v}_\alpha^n)}{\partial(\widetilde{v}_\beta^1, \ldots, \widetilde{v}_\beta^n)} = \left| \frac{\varphi_\beta}{\varphi_\alpha} \right| > 0 \quad \text{for any } \alpha, \beta.$$

Therefore, M is orientable. □

The significance of Proposition 4.25 is that it relates the orientability of an n-manifold (which was defined in a rather *local* way) with the existence of a non-vanishing n-form (which is a *global* object). For abstract manifolds, unit normal vectors cannot be defined. Here the non-vanishing global n-form plays a similar role as a continuous unit normal does for hypersurfaces. In the rest of the course we will call:

Definition 4.26 (Orientation of Manifolds)**.** Given an orientable manifold M^n, a non-vanishing global n-form Ω is called an *orientation of* M. A basis of tangent vectors $\{T_1, \ldots, T_n\} \in T_pM$ is said to be Ω*-oriented* if $\Omega(T_1, \ldots, T_n) > 0$. A local coordinate system $(u_1, \ldots, u_n)$ is said to be Ω*-oriented* if $\Omega\left(\frac{\partial}{\partial u_1}, \ldots, \frac{\partial}{\partial u_n}\right) > 0$.

Recall that when we integrate an n-form, we need to first pick an order of local coordinates $(u_1, \ldots, u_n)$, then express the n-form according to this order, and locally define the integral as:

$$\int_{\mathsf{F}(\mathcal{U})} \varphi\, du^1 \wedge \cdots \wedge du^n = \int_{\mathcal{U}} \varphi\, du^1 \cdots du^n.$$

Note that picking the *order of coordinates* is a *local* notion. To rephrase it using *global* terms, we can first pick an orientation Ω (which is a global object on M), then we require the order of any local coordinates $(u_1, \ldots, u_n)$ to be Ω-oriented. Any pair of local coordinate systems $(u_1, \ldots, u_n)$ and $(v_1, \ldots, v_n)$ which are both Ω-oriented will automatically satisfy $\det \dfrac{\partial(u_1, \ldots, u_n)}{\partial(v_1, \ldots, v_n)} > 0$ on the overlap.

To summarize, given an orientable manifold M^n with a chosen orientation Ω, then for *any* local coordinate system $\mathsf{F}(u_1, \ldots, u_n) : \mathcal{U} \to M$, we define:

$$\int_{\mathsf{F}(\mathcal{U})} \varphi\, du^1 \wedge \cdots \wedge du^n = \begin{cases} \int_{\mathcal{U}} \varphi\, du^1 \cdots du^n & \text{if } (u_1, \ldots, u_n) \text{ is } \Omega\text{-oriented} \\ -\int_{\mathcal{U}} \varphi\, du^1 \cdots du^n & \text{if } (u_1, \ldots, u_n) \text{ is not } \Omega\text{-oriented} \end{cases}$$

or to put it in a more elegant (yet equivalent) way:

$$\int_{\mathsf{F}(\mathcal{U})} \varphi\, du^1 \wedge \cdots \wedge du^n = \operatorname{sgn}\left[\Omega\left(\frac{\partial}{\partial u_1}, \ldots, \frac{\partial}{\partial u_n}\right)\right] \int_{\mathcal{U}} \varphi\, du^1 \cdots du^n.$$

Exercise 4.14. Let $\Omega := dx \wedge dy \wedge dz$ be the orientation of $\mathbb{R}^3$. Which of the following is Ω-oriented?

(a) local coordiantes (x, y, z)

(b) vectors $\{\mathsf{i}, \mathsf{k}, \mathsf{j}\}$

(c) vectors $\{\mathsf{u}, \mathsf{v}, \mathsf{u} \times \mathsf{v}\}$ where u and v are linearly independent vectors in $\mathbb{R}^3$.

Exercise 4.15. Consider three linearly independent vectors $\{\mathsf{u}, \mathsf{v}, \mathsf{w}\}$ in $\mathbb{R}^3$ such that $\mathsf{u} \perp \mathsf{w}$ and $\mathsf{v} \perp \mathsf{w}$. Show that $\{\mathsf{u}, \mathsf{v}, \mathsf{w}\}$ has the same orientation as $\{\mathsf{i}, \mathsf{j}, \mathsf{k}\}$ if and only if $\mathsf{w} = c\mathsf{u} \times \mathsf{v}$ for some *positive* constant c.

4.4. Generalized Stokes' Theorem

In this section, we (finally) state and give a proof of an elegant theorem, Generalized Stokes' Theorem. It not only unifies Green's, Stokes' and Divergence Theorems which we learned in Multivariable Calculus, but also generalize it to higher dimensional abstract manifolds.

4.4.1. Boundary Orientation. Since the statement of Generalized Stokes' Theorem involves integration on differential forms, we will assume all manifolds discussed in this section to be *orientable*. Let's fix an orientation Ω of M^n, which is a non-vanishing n-form, and this orientation determines how local coordinates on M are ordered as discussed in the previous section.

Now we deal with the orientation of the boundary manifold ∂M. Given a local parametrization $\mathsf{G}(u_1, \ldots, u_n) : \mathcal{V} \subset \mathbb{R}^n_+ \to M$ of boundary type. The tangent space T_pM for points $p \in \partial M$ is defined as the span of $\left\{ \dfrac{\partial}{\partial u_i} \right\}_{i=1}^n$. As $\mathcal{V}$ is a subset of the upper half-space $\{u_n \geq 0\}$, the vector $\nu := -\frac{\partial}{\partial u_n}$ in T_pM is often called an *outward-pointing* "normal" vector to ∂M.

An orientation Ω of M^n is a non-vanishing n-form. The boundary manifold ∂M^n is an $(n-1)$-manifold, and so an orientation of ∂M^n should be a non-vanishing $(n-1)$-form. Using the outward-pointing normal vector ν, one can produce such an $(n-1)$-form in a natural way. Given any tangent vectors $T_1, \ldots, T_{n-1}$ on $T(\partial M)$, we define the following multilinear map:

$$(i_\nu \Omega)(T_1, \ldots, T_{n-1}) := \Omega(\nu, T_1, \ldots, T_{n-1}).$$

Then $i_\nu \Omega$ is an alternating multilinear map in $\wedge^{n-1} T^*(\partial M)$.

Locally, given a local coordinate system $(u_1, \ldots, u_n)$, by recalling that $\nu = -\frac{\partial}{\partial u_n}$ we can compute:

$$\begin{aligned}
(i_\nu \Omega) \left(\frac{\partial}{\partial u_1}, \ldots, \frac{\partial}{\partial u_{n-1}} \right) &= \Omega \left(\nu, \frac{\partial}{\partial u_1}, \ldots, \frac{\partial}{\partial u_{n-1}} \right) \\
&= \Omega \left(-\frac{\partial}{\partial u_n}, \frac{\partial}{\partial u_1}, \ldots, \frac{\partial}{\partial u_{n-1}} \right) \\
&= (-1)^n \Omega \left(\frac{\partial}{\partial u_1}, \ldots, \frac{\partial}{\partial u_{n-1}}, \frac{\partial}{\partial u_n} \right)
\end{aligned}$$

which is non-zero. Therefore, $i_\nu \Omega$ is a non-vanishing $(n-1)$-form on ∂M, and we can take it as an orientation for ∂M. From now on, whenever we pick an orientation Ω for M^n, we will by-default pick $i_\nu \Omega$ to be the orientation for ∂M.

Given an Ω-oriented local coordinate system $\mathsf{G}(u_1, \ldots, u_n) : \mathcal{V} \to M$ of boundary type for M^n, then $(u_1, \ldots, u_{n-1})$ is $i_\nu \Omega$-oriented if n is *even*; and is not $i_\nu \Omega$-oriented if n is *odd*. Therefore, when integrating an $(n-1)$-form $\varphi\, du^1 \wedge \cdots \wedge du^{n-1}$ on ∂M, we need to take into account of the parity of n, i.e.

$$\int_{\mathsf{G}(\mathcal{V}) \cap \partial M} \varphi\, du^1 \wedge \cdots \wedge du^{n-1} = (-1)^n \int_{\mathcal{V} \cap \{u_n = 0\}} \varphi\, du^1 \cdots du^{n-1}. \tag{4.6}$$

The "extra" factor of $(-1)^n$ does not look nice at the first glance, but as we will see later, it will make Generalized Stokes' Theorem nicer. We are now ready to state Generalized Stokes' Theorem in a precise way:

Theorem 4.27 (Generalized Stokes' Theorem)**.** *Let M be an orientable smooth n-manifold, and let ω be a compactly supported smooth $(n-1)$-form on M. Then, we have:*

$$\int_M d\omega = \int_{\partial M} \omega. \tag{4.7}$$

Here if Ω is a chosen to be an orientation of M, then we will take $i_\nu\Omega$ to be the orientation of ∂M where ν is an outward-point normal vector of ∂M.

In particular, if $\partial M = \emptyset$, then $\int_M d\omega = 0$.

4.4.2. Proof of Generalized Stokes' Theorem. The proof consists of three steps:

Step 1: a special case where $\operatorname{supp}\omega$ is contained inside a single parametrization chart of interior type;

Step 2: another special case where $\operatorname{supp}\omega$ is contained inside a single parametrization chart of boundary type;

Step 3: use partitions of unity to deduce the general case.

Proof of Theorem 4.27. Throughout the proof, we will let Ω be the orientation of M, and $i_\nu\Omega$ be the orientation of ∂M with ν being an outward-point normal vector to ∂M. All local coordinate system $(u_1, \ldots, u_n)$ of M is assumed to be Ω-oriented.

Step 1: Suppose $\operatorname{supp}\omega$ *is contained in a single parametrization chart of interior type.*

Let $\mathsf{F}(u_1, \ldots, u_n) : \mathcal{U} \subset \mathbb{R}^n \to M$ be a local parametrization of interior type such that $\operatorname{supp}\omega \subset \mathsf{F}(\mathcal{U})$. Denote:

$$du^1 \wedge \cdots \wedge \widehat{du^i} \wedge \cdots \wedge du^n := du^1 \wedge \cdots \wedge du^{i-1} \wedge du^{i+1} \wedge \cdots \wedge du^n,$$

or in other words, it means the form with du^i removed.

In terms of local coordinates, the $(n-1)$-form ω can be expressed as:

$$\omega = \sum_{i=1}^n \omega_i \, du^1 \wedge \cdots \wedge \widehat{du^i} \wedge \cdots \wedge du^n.$$

Taking the exterior derivative, we get:

$$d\omega = \sum_{i=1}^n \sum_{j=1}^n \frac{\partial \omega_i}{\partial u_j} du^j \wedge du^1 \wedge \cdots \wedge \widehat{du^i} \wedge \cdots \wedge du^n$$

For each i, the wedge product $du^j \wedge du^1 \wedge \cdots \wedge \widehat{du^i} \wedge \cdots \wedge du^n$ is zero if $j \neq i$. Therefore,

$$\begin{aligned} d\omega &= \sum_{i=1}^n \frac{\partial \omega_i}{\partial u_i} du^i \wedge du^1 \wedge \cdots \wedge \widehat{du^i} \wedge \cdots \wedge du^n \\ &= \sum_{i=1}^n (-1)^{i-1} \frac{\partial \omega_i}{\partial u_i} du^1 \wedge \cdots \wedge du^i \wedge \cdots \wedge du^n \end{aligned}$$

By definition of integrals of differential forms, we get:

$$\int_M d\omega = \int_{\mathcal{U}} \sum_{i=1}^n (-1)^{i-1} \frac{\partial \omega_i}{\partial u_i} du^1 \cdots du^n.$$

Since $\operatorname{supp}\omega \subset \mathsf{F}(\mathcal{U})$, the functions ω_i's are identically zero near and outside the boundary of $\mathcal{U} \subset \mathbb{R}^n$. Therefore, we can replace the domain of integration $\mathcal{U}$ of the RHS integral by a rectangle $[-R, R] \times \cdots \times [-R, R]$ in $\mathbb{R}^n$ where $R > 0$ is a sufficiently large number. The value of the integral is unchanged. Therefore, using the Fubini's Theorem, we get:

$$\begin{aligned}\int_M d\omega &= \int_{-R}^{R} \cdots \int_{-R}^{R} \sum_{i=1}^{n} (-1)^i \frac{\partial \omega_i}{\partial u_i} du^1 \cdots du^n \\ &= \sum_{i=1}^{n} (-1)^{i-1} \int_{-R}^{R} \cdots \int_{-R}^{R} \frac{\partial \omega_i}{\partial u_i} du^i \, du^1 \cdots \widehat{du^i} \cdots du^n \\ &= \sum_{i=1}^{n} (-1)^{i-1} \int_{-R}^{R} \cdots \int_{-R}^{R} [\omega_i]_{u_i=-R}^{u_i=R} \, du^1 \cdots \widehat{du^i} \cdots du^n.\end{aligned}$$

Since ω_i's vanish at the boundary of the rectangle $[-R, R]^n$, we have $\omega_i = 0$ when $u_i = \pm R$. As a result, we proved $\int_M d\omega = 0$. Since $\operatorname{supp} \omega$ is contained in a single parametrization chart of interior type, we have $\omega = 0$ on the boundary ∂M. Evidently, we have $\int_{\partial M} \omega = 0$ in this case. Hence, we proved

$$\int_M d\omega = \int_{\partial M} \omega = 0$$

in this case.

Step 2: Suppose $\operatorname{supp} \omega$ *is contained inside a single parametrization chart of boundary type.*

Let $\mathsf{G}(u_1, \ldots, u_n) : \mathcal{V} \subset \mathbb{R}_+^n \to M$ be a local parametrization of boundary type such that $\operatorname{supp} \omega \subset \mathsf{G}(\mathcal{V})$. As in Step 1, we express

$$\omega = \sum_{i=1}^{n} \omega_i \, du^1 \wedge \cdots \wedge \widehat{du^i} \wedge \cdots \wedge du^n.$$

Proceed exactly in the same way as before, we arrive at:

$$\int_M d\omega = \int_{\mathcal{V}} \sum_{i=1}^{n} (-1)^{i-1} \frac{\partial \omega_i}{\partial u_i} du^1 \cdots du^n.$$

Now $\mathcal{V}$ is an open set in $\mathbb{R}_+^n$ instead of $\mathbb{R}^n$. Recall that the boundary is the set of points with $u_n = 0$. Therefore, this time we replace $\mathcal{V}$ by the half-space rectangle $[-R, R] \times \cdots \times [-R, R] \times [0, R]$ where $R > 0$ again is a sufficiently large number.

One key difference from Step 1 is that even though ω_i's has compact support inside $\mathcal{V}$, it may not vanish on the boundary of M. Therefore, we can only guarantee $\omega_i(u_1, \ldots, u_n) = 0$ when $u_n = R$, but we cannot claim $\omega_i = 0$ when $u_n = 0$. Some more work needs to be done:

$$\begin{aligned}\int_M d\omega &= \int_{\mathcal{V}} \sum_{i=1}^{n} (-1)^{i-1} \frac{\partial \omega_i}{\partial u_i} du^1 \cdots du^n \\ &= \int_0^R \int_{-R}^{R} \cdots \int_{-R}^{R} \sum_{i=1}^{n} (-1)^{i-1} \frac{\partial \omega_i}{\partial u_i} du^1 \cdots du^n \\ &= \sum_{i=1}^{n-1} (-1)^{i-1} \int_0^R \int_{-R}^{R} \cdots \int_{-R}^{R} (-1)^{i-1} \frac{\partial \omega_i}{\partial u_i} du^1 \cdots du^n \\ &\quad + (-1)^{n-1} \int_0^R \int_{-R}^{R} \cdots \int_{-R}^{R} \frac{\partial \omega_n}{\partial u_n} du^1 \cdots du^n\end{aligned}$$

One can proceed as in Step 1 to show that the first term:

$$\sum_{i=1}^{n-1} (-1)^{i-1} \int_0^R \int_{-R}^{R} \cdots \int_{-R}^{R} (-1)^{i-1} \frac{\partial \omega_i}{\partial u_i} du^1 \cdots du^n = 0,$$

which follows from the fact that whenever $1 \leq i \leq n-1$, we have $\omega_i = 0$ on $u_i = \pm R$.

The second term:

$$(-1)^{n-1} \int_0^R \int_{-R}^R \cdots \int_{-R}^R \frac{\partial \omega_n}{\partial u_n} du^1 \cdots du^n$$

is handled in a different way:

$$\begin{aligned}
&(-1)^{n-1} \int_0^R \int_{-R}^R \cdots \int_{-R}^R \frac{\partial \omega_n}{\partial u_n} du^1 \cdots du^n \\
&= (-1)^{n-1} \int_{-R}^R \cdots \int_{-R}^R \int_0^R \frac{\partial \omega_n}{\partial u_n}\, du^n\, du^1 \cdots du^{n-1} \\
&= (-1)^{n-1} \int_{-R}^R \cdots \int_{-R}^R [\omega_n]_{u_n=0}^{u_n=R}\, du^1 \cdots du^{n-1} \\
&= (-1)^n \int_{-R}^R \cdots \int_{-R}^R \omega_n(u_1, \ldots, u_{n-1}, 0)\, du^1 \cdots du^{n-1}
\end{aligned}$$

where we have used the following fact:

$$\begin{aligned}
[\omega_n(u_1, \ldots, u_n)]_{u_n=0}^{u_n=R} &= \omega_n(u_1, \ldots, u_{n-1}, R) - \omega_i(u_1, \ldots, u_{n-1}, 0) \\
&= 0 - \omega_n(u_1, \ldots, u_{n-1}, 0).
\end{aligned}$$

Combining all results proved so far, we have:

$$\int_M d\omega = (-1)^n \int_{-R}^R \cdots \int_{-R}^R \omega_n(u_1, \ldots, u_{n-1}, 0)\, du^1 \cdots du^{n-1}$$

On the other hand, we compute $\int_{\partial M} \omega$ and then compare it with $\int_M d\omega$. Note that the boundary ∂M are points with $u_n = 0$. Therefore, across the boundary ∂M, we have $du^n \equiv 0$, and so on ∂M we have:

$$\begin{aligned}
\omega &= \sum_{i=1}^n \omega_i(u_1, \ldots, u_{n-1}, 0)\, du^1 \wedge \cdots \wedge \widehat{du^i} \wedge \cdots \wedge \underbrace{du^n}_{=0} \\
&= \omega_n(u_1, \ldots, u_{n-1}, 0)\, du^1 \wedge \cdots \wedge du^{n-1} \\
\int_{\partial M} \omega &= \int_{\mathsf{G}(\mathcal{V}) \cap \partial M} \omega_n(u_1, \ldots, u_{n-1}, 0)\, du^1 \wedge \cdots \wedge du^{n-1} \\
&= (-1)^n \int_{\mathcal{V} \cap \{u_n = 0\}} \omega_n(u_1, \ldots, u_{n-1}, 0)\, du^1 \cdots du^{n-1} \\
&= (-1)^n \int_{-R}^R \cdots \int_{-R}^R \omega_n(u_1, \ldots, u_{n-1}, 0)\, du^1 \cdots du^{n-1}
\end{aligned}$$

Recall that we have a factor of $(-1)^n$ because the local coordinate system $(u_1, \ldots, u_{n-1})$ for ∂M is $i_\nu \Omega$ if and only if n is even, as discussed in the previous subsection.

Consequently, we have proved

$$\int_M d\omega = \int_{\partial M} \omega$$

in this case.

Step 3: Use partitions of unity to deduce the general case

Finally, we "glue" the previous two steps together and deduce the general case. Let $\mathcal{A} = \{\mathsf{F}_\alpha : \mathcal{U}_\alpha \to M\}$ be an atlas of M where all local coordinates are Ω-oriented. Here $\mathcal{A}$ contain both interior and boundary types of local parametrizations. Suppose $\{\rho_\alpha : M \to [0,1]\}$ is a partition of unity subordinate to $\mathcal{A}$. Then, we have:

$$\omega = \underbrace{\left(\sum_\alpha \rho_\alpha\right)}_{\equiv 1} \omega = \sum_\alpha \rho_\alpha \omega$$

$$\int_{\partial M} \omega = \int_{\partial M} \sum_\alpha \rho_\alpha \omega = \sum_\alpha \int_{\partial M} \rho_\alpha \omega.$$

For each α, the $(n-1)$-form $\rho_\alpha \omega$ is compactly supported in a single parametrization chart (either of interior or boundary type). From Step 1 and Step 2, we have already proved that Generalized Stokes' Theorem is true for each $\rho_\alpha \omega$. Therefore, we have:

$$\begin{aligned}\sum_\alpha \int_{\partial M} \rho_\alpha \omega &= \sum_\alpha \int_M d(\rho_\alpha \omega)\\ &= \sum_\alpha \int_M (d\rho_\alpha \wedge \omega + \rho_\alpha \, d\omega)\\ &= \int_M d\left(\sum_\alpha \rho_\alpha\right) \wedge \omega + \left(\sum_\alpha \rho_\alpha\right) d\omega.\end{aligned}$$

Since $\sum_\alpha \rho_\alpha \equiv 1$ and hence $d\left(\sum_\alpha \rho_\alpha\right) \equiv 0$, we have proved:

$$\int_{\partial M} \omega = \sum_\alpha \int_{\partial M} \rho_\alpha \omega = \int_M 0 \wedge \omega + 1\, d\omega = \int_M d\omega.$$

It completes the proof of Generalized Stokes' Theorem. □

Remark 4.28. As we can see from that the proof (Step 2), if we simply choose an orientation for ∂M such that $(u_1, \ldots, u_{n-1})$ becomes the order of local coordinates for ∂M, then (4.7) would have a factor of $(-1)^n$ on the RHS, which does not look nice. Moreover, if we pick $i_{-\nu}\Omega$ to be the orientation of ∂M (here $-\nu$ is then an inward-pointing normal to ∂M), then the RHS of (4.7) would have a minus sign, which is not nice either. □

4.4.3. Fundamental Theorems in Vector Calculus. We briefly discussed at the end of Chapter 3 how the three fundamental theorems in Vector Calculus, namely Green's, Stokes' and Divergence Theorems, can be formulated using differential forms. Given that we now have proved Generalized Stokes' Theorem (Theorem 4.27), we are going to give a formal proof of the three Vector Calculus theorems in MATH 2023 using the Theorem 4.27.

Corollary 4.29 (Green's Theorem). *Let R be a closed and bounded smooth 2-submanifold in $\mathbb{R}^2$ with boundary ∂R. Given any smooth vector field $\mathsf{V} = (P(x,y), Q(x,y))$ defined in R, then we have:*

$$\oint_{\partial R} \mathsf{V} \cdot d\mathsf{r} = \int_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\, dy,$$

The line integral on the LHS is oriented such that $\{\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\}$ has the same orientation as $\{\nu, \mathsf{T}\}$ where ν is the outward-pointing normal of R, and T is the velocity vector of the curve ∂R. See Figure 4.3.

Proof. Consider the 1-form $\omega := P\, dx + Q\, dy$ defined on R, then we have:

$$d\omega = \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx \wedge dy.$$

Suppose we fix an orientation $\Omega = dx \wedge dy$ for R so that the order of coordinates is (x, y), then by generalized Stokes' Theorem we get:

$$\underbrace{\oint_{\partial R} P\,dx + Q\,dy}_{\oint_{\partial R} \omega} = \underbrace{\int_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx \wedge dy}_{\int_R d\omega} = \underbrace{\int_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy}_{(x,y) \text{ is the orientation}}.$$

The only thing left to figure out is the orientation of the line integral. Locally parametrize R by local coordinates (s, t) so that $\{t = 0\}$ is the boundary ∂R and $\{t > 0\}$ is the interior of R (see Figure 4.3). By convention, the local coordinate s for ∂R must be chosen so that $\Omega(\nu, \frac{\partial}{\partial s}) > 0$ where ν is a outward-pointing normal vector to ∂R. In other words, the pair $\{\nu, \frac{\partial}{\partial s}\}$ should have the same orientation as $\{\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\}$. According to Figure 4.3, we must choose the local coordinate s for ∂R such that for the outer boundary, s goes counter-clockwisely as it increases; whereas for each inner boundary, s goes clockwisely as it increases. □

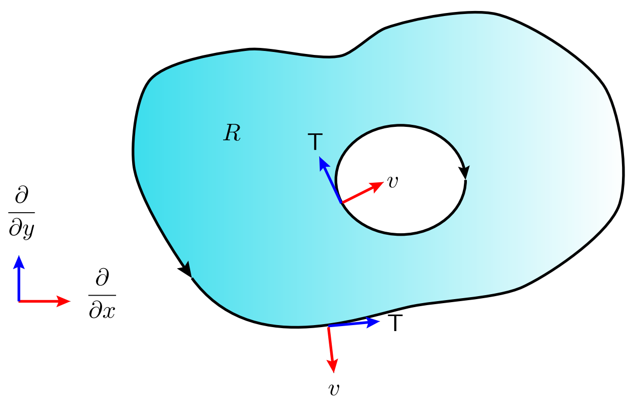

Figure 4.3. Orientation of Green's Theorem

Next we show that Stokes' Theorem in Multivariable Calculus is also a consequence of Generalized Stokes' Theorem. Recall that in MATH 2023 we learned about surface integrals. If $\mathsf{F}(u, v) : \mathcal{U} \to \Sigma \subset \mathbb{R}^3$ is a parametrization of the whole surface Σ, then we define the surface element as:

$$dS = \left|\frac{\partial \mathsf{F}}{\partial u} \times \frac{\partial \mathsf{F}}{\partial v}\right| du\,dv,$$

and the surface integral of a scalar function φ is defined as:

$$\int_\Sigma \varphi\,dS = \int_{\mathcal{U}} \varphi(u, v) \left|\frac{\partial \mathsf{F}}{\partial u} \times \frac{\partial \mathsf{F}}{\partial v}\right| du\,dv.$$

However, not every surface can be covered (or almost covered) by a single parametrization chart. Generally, if $\mathcal{A} = \{\mathsf{F}_\alpha(u_\alpha, v_\alpha) : \mathcal{U}_\alpha \to \mathbb{R}^3\}$ is an oriented atlas of Σ with a partition of unity $\{\rho_\alpha : \Sigma \to [0, 1]\}$ subordinate to $\mathcal{A}$, we then define:

$$dS := \sum_\alpha \rho_\alpha \left|\frac{\partial \mathsf{F}_\alpha}{\partial u_\alpha} \times \frac{\partial \mathsf{F}_\alpha}{\partial v_\alpha}\right| du_\alpha\,dv_\alpha.$$

Corollary 4.30 (Stokes' Theorem)**.** *Let Σ be a closed and bounded smooth 2-submanifold in $\mathbb{R}^3$ with boundary $\partial\Sigma$, and $\mathsf{V} = (P(x,y,z), Q(x,y,z), R(x,y,z))$ be a vector field which is smooth on Σ, then we have:*

$$\oint_{\partial\Sigma} \mathsf{V}\cdot d\mathsf{r} = \int_{\Sigma} (\nabla\times\mathsf{V})\cdot\mathsf{N}\,dS.$$

Here $\{\mathsf{i},\mathsf{j},\mathsf{k}\}$ has the same orientation as $\{\nu,\mathsf{T},\mathsf{N}\}$, where ν is the outward-point normal vector of Σ at points of $\partial\Sigma$, T is the velocity vector of $\partial\Sigma$, and N is the unit normal vector to Σ in $\mathbb{R}^3$. See Figure 4.4.

Proof. Define:

$$\omega = P\,dx + Q\,dy + R\,dz$$

which is viewed as a 1-form on Σ. Then,

$$\oint_{\partial\Sigma} \omega = \oint_{\partial\Sigma} \mathsf{V}\cdot d\mathsf{r}. \tag{4.8}$$

By direct computation, the 2-form $d\omega$ is given by:

$$d\omega = \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\wedge dy + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right) dz\wedge dx + \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right) dy\wedge dz.$$

Now consider an oriented atlas $\mathcal{A} = \{\mathsf{F}_\alpha(u_\alpha, v_\alpha) : \mathcal{U}_\alpha \to \mathbb{R}^3\}$ of Σ with a partition of unity $\{\rho_\alpha : \Sigma \to [0,1]\}$, then according to the discussion near the end of Chapter 3, we can express each of $dx\wedge dy$, $dz\wedge dx$ and $dy\wedge dz$ in terms of $du_\alpha \wedge dv_\alpha$, and obtain:

$$\begin{aligned}
d\omega &= \sum_\alpha \rho_\alpha\, d\omega \\
&= \sum_\alpha \rho_\alpha \left[\left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\wedge dy + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right) dz\wedge dx + \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right) dy\wedge dz\right] \\
&= \sum_\alpha \rho_\alpha \left\{\left(\left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\det\frac{\partial(y,z)}{\partial(u_\alpha,v_\alpha)} + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right)\det\frac{\partial(z,x)}{\partial(u_\alpha,v_\alpha)}\right.\right. \\
&\qquad \left.\left. + \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right)\det\frac{\partial(x,y)}{\partial(u_\alpha,v_\alpha)}\right)\right\} du_\alpha\wedge dv_\alpha.
\end{aligned}$$

On each local coordinate chart $\mathsf{F}_\alpha(\mathcal{U}_\alpha)$, a normal vector to Σ in $\mathbb{R}^3$ can be found using cross products:

$$\begin{aligned}
\frac{\partial \mathsf{F}_\alpha}{\partial u_\alpha} \times \frac{\partial \mathsf{F}_\alpha}{\partial v_\alpha} &= \det\frac{\partial(y,z)}{\partial(u_\alpha,v_\alpha)}\mathsf{i} + \det\frac{\partial(z,x)}{\partial(u_\alpha,v_\alpha)}\mathsf{j} + \det\frac{\partial(x,y)}{\partial(u_\alpha,v_\alpha)}\mathsf{k} \\
\nabla\times\mathsf{V} &= \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\mathsf{i} + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right)\mathsf{j} + \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right)\mathsf{k}.
\end{aligned}$$

Hence,

$$d\omega = \sum_\alpha (\nabla\times\mathsf{V})\cdot\left(\frac{\partial \mathsf{F}_\alpha}{\partial u_\alpha} \times \frac{\partial \mathsf{F}_\alpha}{\partial v_\alpha}\right)\rho_\alpha\, du_\alpha\wedge dv_\alpha,$$

and so

$$\int_\Sigma d\omega = \sum_\alpha \int_{\mathcal{U}_\alpha} (\nabla\times\mathsf{V})\cdot\left(\frac{\partial \mathsf{F}_\alpha}{\partial u_\alpha} \times \frac{\partial \mathsf{F}_\alpha}{\partial v_\alpha}\right)\rho_\alpha\, du_\alpha\, dv_\alpha.$$

Denote $\mathsf{N} = \dfrac{\frac{\partial \mathsf{F}_\alpha}{\partial u_\alpha} \times \frac{\partial \mathsf{F}_\alpha}{\partial v_\alpha}}{\left|\frac{\partial \mathsf{F}}{\partial u_\alpha} \times \frac{\partial \mathsf{F}}{\partial v_\alpha}\right|}$, and recall the fact that $dS := \sum_\alpha \rho_\alpha \left|\dfrac{\partial \mathsf{F}_\alpha}{\partial u_\alpha} \times \dfrac{\partial \mathsf{F}_\alpha}{\partial v_\alpha}\right| du_\alpha\, dv_\alpha$, we get:

$$\int_\Sigma d\omega = \int_\Sigma (\nabla\times\mathsf{V})\cdot\mathsf{N}\,dS. \tag{4.9}$$

Combining the results of (4.8) and (4.9), using Generalized Stokes' Theorem (Theorem 4.7, we get:

$$\oint_{\partial\Sigma} \mathsf{V} \cdot d\mathsf{r} = \int_{\Sigma} (\nabla \times \mathsf{V}) \cdot \mathsf{N}\, dS$$

as desired.

To see the orientation of $\partial\Sigma$, we locally parametrize Σ by coordinates (s, t) such that $\{t = 0\}$ are points on $\partial\Sigma$, and so $\partial\Sigma$ is locally parametrized by s. The outward-pointing normal of $\partial\Sigma$ in Σ is given by $\nu := -\frac{\partial}{\partial t}$. By convention, the orientation of $\left\{\nu, \frac{\partial}{\partial s}\right\}$ is the same as $\left\{\frac{\partial}{\partial u_\alpha}, \frac{\partial}{\partial v_\alpha}\right\}$, and hence:

$$\left\{\nu, \frac{\partial}{\partial s}, \mathsf{N}\right\} \text{ has the same orientation as } \left\{\frac{\partial}{\partial u_\alpha}, \frac{\partial}{\partial v_\alpha}, \mathsf{N}\right\}.$$

As $\mathsf{N} = \dfrac{\frac{\partial \mathsf{F}_\alpha}{\partial u_\alpha} \times \frac{\partial \mathsf{F}_\alpha}{\partial v_\alpha}}{\left|\frac{\partial \mathsf{F}_\alpha}{\partial u_\alpha} \times \frac{\partial \mathsf{F}_\alpha}{\partial v_\alpha}\right|}$, the set $\left\{\frac{\partial}{\partial u_\alpha}, \frac{\partial}{\partial v_\alpha}, \mathsf{N}\right\}$ has the same orientation as $\{\mathsf{i}, \mathsf{j}, \mathsf{k}\}$. As a result, the set $\{\nu, \frac{\partial}{\partial s}, \mathsf{N}\}$ is oriented in the way as in Figure 4.4. □

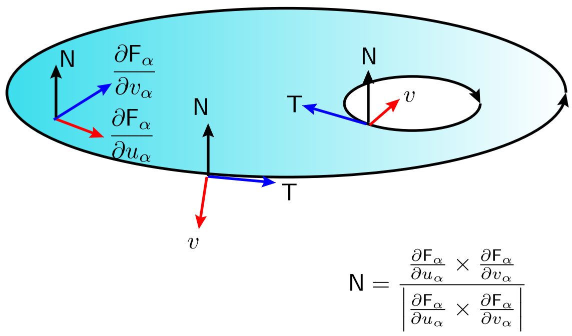

Figure 4.4. Orientation of Stokes' Theorem

Finally, we discuss how to use Generalized Stokes' Theorem to prove Divergence Theorem in Multivariable Calculus.

Corollary 4.31 (Divergence Theorem)**.** *Let D be a closed and bounded 3-submanifold of $\mathbb{R}^3$ with boundary ∂D, and $\mathsf{V} = (P(x,y,z), Q(x,y,z), R(x,y,z))$ be a smooth vector field defined on D. Then, we have:*

$$\oint_{\partial D} \mathsf{V} \cdot \mathsf{N}\, dS = \int_D \nabla \cdot \mathsf{V}\, dx\, dy\, dz.$$

Here N is the unit normal vector of ∂D in $\mathbb{R}^3$ which points away from D.

Proof. Let $\omega := P\, dy \wedge dz + Q\, dz \wedge dx + R\, dx \wedge dy$. Then by direct computations, we get:

$$d\omega = \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}\right) dx \wedge dy \wedge dz = \nabla \cdot \mathsf{V}\, dx \wedge dy \wedge dz.$$

Using $\{\mathsf{i}, \mathsf{j}, \mathsf{k}\}$ as the orientation for D, then it is clear that:

$$\int_D d\omega = \int_D \nabla \cdot \mathsf{V}\, dx\, dy\, dz. \tag{4.10}$$

Consider an atlas $\mathcal{A} = \{\mathsf{F}_\alpha(u_\alpha, v_\alpha, w_\alpha) : \mathcal{U}_\alpha \to \mathbb{R}^3\}$ of D such that for the local parametrization of boundary type, the boundary points are given by $\{w_\alpha = 0\}$, and interior points are $\{w_\alpha > 0\}$. Then, ∂D is locally parametrized by (u_α, v_α).

As a convention, the orientation of (u_α, v_α) is chosen such that $\{-\frac{\partial}{\partial w_\alpha}, \frac{\partial}{\partial u_\alpha}, \frac{\partial}{\partial v_\alpha}\}$ has the same orientation as $\{\mathsf{i}, \mathsf{j}, \mathsf{k}\}$, or equivalently, $\{\frac{\partial}{\partial u_\alpha}, \frac{\partial}{\partial v_\alpha}, -\frac{\partial}{\partial w_\alpha}\}$ has the same orientation as $\{\mathsf{i}, \mathsf{j}, \mathsf{k}\}$.

Furthermore, let N be the unit normal of ∂D given by:

$$\mathsf{N} = \frac{\frac{\partial \mathsf{F}_\alpha}{\partial u_\alpha} \times \frac{\partial \mathsf{F}_\alpha}{\partial v_\alpha}}{\left|\frac{\partial \mathsf{F}_\alpha}{\partial u_\alpha} \times \frac{\partial \mathsf{F}_\alpha}{\partial v_\alpha}\right|}.$$

By the convention of cross products, $\{\frac{\partial \mathsf{F}_\alpha}{\partial u_\alpha}, \frac{\partial \mathsf{F}_\alpha}{\partial v_\alpha}, \mathsf{N}\}$ must have the same orientation as $\{\mathsf{i}, \mathsf{j}, \mathsf{k}\}$. Now that $\{\frac{\partial}{\partial u_\alpha}, \frac{\partial}{\partial v_\alpha}, -\frac{\partial}{\partial w_\alpha}\}$ and $\{\frac{\partial \mathsf{F}_\alpha}{\partial u_\alpha}, \frac{\partial \mathsf{F}_\alpha}{\partial v_\alpha}, \mathsf{N}\}$ have the same orientation, so N and $-\frac{\partial}{\partial w_\alpha}$ are both pointing in the same direction. In other words, N is the outward-point normal.

The rest of the proof goes by writing ω in terms of $du_\alpha \wedge dv_\alpha$ on each local coordinate chart:

$$\begin{aligned}
\omega &= \sum_\alpha \rho_\alpha \omega \\
&= \sum_\alpha \rho_\alpha \left(P \det \frac{\partial(y,z)}{\partial(u_\alpha, v_\alpha)} + Q \det \frac{\partial(z,x)}{\partial(u_\alpha, v_\alpha)} + R \det \frac{\partial(x,y)}{\partial(u_\alpha, v_\alpha)} \right) du_\alpha \wedge dv_\alpha \\
&= \sum_\alpha \mathsf{V} \cdot \left(\frac{\partial \mathsf{F}_\alpha}{\partial u_\alpha} \times \frac{\partial \mathsf{F}_\alpha}{\partial v_\alpha} \right) \rho_\alpha \, du_\alpha \wedge dv_\alpha \\
&= \sum_\alpha \mathsf{V} \cdot \mathsf{N} \, \rho_\alpha \left| \frac{\partial \mathsf{F}_\alpha}{\partial u_\alpha} \times \frac{\partial \mathsf{F}_\alpha}{\partial v_\alpha} \right| du_\alpha \wedge dv_\alpha
\end{aligned}$$

Therefore, we get:

$$\oint_{\partial D} \omega = \oint_{\partial D} \sum_\alpha \mathsf{V} \cdot \mathsf{N} \, \rho_\alpha \left| \frac{\partial \mathsf{F}_\alpha}{\partial u_\alpha} \times \frac{\partial \mathsf{F}_\alpha}{\partial v_\alpha} \right| du_\alpha \, dv_\alpha = \oint_{\partial D} \mathsf{V} \cdot \mathsf{N} \, dS. \tag{4.11}$$

Combining with (4.10), (4.11) and Generalized Stokes' Theorem, the proof of this corollary is completed. □