Chapter 3

Tensors and Differential Forms

"In the beginning, God said, the four-dimensional divergence of an antisymmetric, second-rank tensor equals zero, and there was light."

Michio Kaku

In Multivariable Calculus, we learned about gradient, curl and divergence of a vector field, and three important theorems associated to them, namely Green's, Stokes' and Divergence Theorems. In this and the next chapters, we will generalize these theorems to higher dimensional manifolds, and unify them into one single theorem (called the *Generalized Stokes' Theorem*). In order to carry out this generalization and unification, we need to introduce tensors and differential forms. The reasons of doing so are *many-folded*. We will explain it in detail. Meanwhile, one obvious reason is that the curl of a vector field is only defined in $\mathbb{R}^3$ since it uses the cross product. In this chapter, we will develop the language of *differential forms* which will be used in place of gradient, curl, divergence and all that in Multivariable Calculus.

3.1. Cotangent Spaces

3.1.1. Review of Linear Algebra: dual spaces. Let V be an n-dimensional real vector space, and $\mathcal{B} = \{e_1, \ldots, e_n\}$ be a basis for V. The set of all linear maps $T : V \to \mathbb{R}$ from V to the scalar field $\mathbb{R}$ (they are commonly called *linear functionals*) forms a vector space with dimension n. This space is called the *dual space* of V, denoted by V^*.

Associated to the basis $\mathcal{B} = \{e_i\}_{i=1}^n$ for V, there is a basis $\mathcal{B}^* = \{e_i^*\}_{i=1}^n$ for V^*:

$$e_i^*(e_j) = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$

The basis $\mathcal{B}^*$ for V^* (do Exericse 3.1 to verify it is indeed a basis) is called the *dual basis* of V^* with respect to $\mathcal{B}$.

Exercise 3.1. Given that V is a finite-dimensional real vector space, show that:

(a) V^* is a vector space

(b) $\dim V^* = \dim V$

(c) If $\mathcal{B} = \{\mathsf{e}_i\}_{i=1}^n$ is a basis for V, then $\mathcal{B}^* := \{\mathsf{e}_i^*\}_{i=1}^n$ is a basis for V^*.

Given $T \in V^*$ and that $T(\mathsf{e}_i) = a_i$, verify that:

$$T = \sum_{i=1}^n a_i \mathsf{e}_i^*.$$

3.1.2. Cotangent Spaces of Smooth Manifolds. Let M^n be a smooth manifold. Around $p \in M$, suppose there is a local parametrization $\mathsf{F}(u_1, \ldots, u_n)$. Recall that the tangent space T_pM at p is defined as the span of partial differential operators $\left\{\frac{\partial}{\partial u_i}(p)\right\}_{i=1}^n$. The *cotangent space* denoted by T_p^*M is defined as follows:

Definition 3.1 (Cotangent Spaces)**.** Let M^n be a smooth manifold. At every $p \in M$, the *cotangent space of M at p* is the dual space of the tangent space T_pM, i.e.:

$$T_p^*M = \left(T_pM\right)^*.$$

The elements in T_p^*M are called *cotangent vectors* of M at p.

Remark 3.2. Some authors use T_pM^* to denote the cotangent space. Some authors (such as **[Lee13]**) also call *cotangent vectors* as *tangent covectors*. □

Associated to the basis $\mathcal{B}_p = \left\{\frac{\partial}{\partial u_i}(p)\right\}_{i=1}^n$ of T_pM, there is a dual basis $\mathcal{B}_p^* = \left\{(du^1)_p, \ldots, (du^n)_p\right\}$ for T_p^*M, which is defined as follows:

$$(du^i)_p\left(\frac{\partial}{\partial u_j}(p)\right) = \delta_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$

As $(du^i)_p$ is a linear map from T_pM to $\mathbb{R}$, from the above definition we have:

$$(du^i)_p\left(\sum_{j=1}^n a_j \frac{\partial}{\partial u_j}(p)\right) = \sum_{j=1}^n a_j \delta_{ij} = a_i.$$

Occasionally (just for aesthetic purpose), $\left(du^i\right)_p$ can be denoted as $du^i\big|_p$. Moreover, whenever p is clear from the context (or not significant), we may simply write du^i and $\frac{\partial}{\partial u_i}$.

Note that both $\mathcal{B}_p$ and $\mathcal{B}_p^*$ depend on the choice of local coordinates. Suppose $(v_1, \ldots, v_n)$ is another local coordinates around p, then by chain rule we have:

$$\frac{\partial}{\partial v_j} = \sum_{k=1}^n \frac{\partial u_k}{\partial v_j}\frac{\partial}{\partial u_k}$$
$$\frac{\partial}{\partial u_j} = \sum_{k=1}^n \frac{\partial v_k}{\partial u_j}\frac{\partial}{\partial v_k}.$$

We are going to express dv^i in terms of du^j's:

$$\begin{aligned} dv^i\left(\frac{\partial}{\partial u_j}\right) &= dv^i\left(\sum_{k=1}^{n}\frac{\partial v_k}{\partial u_j}\frac{\partial}{\partial v_k}\right) \\ &= \sum_{k=1}^{n}\frac{\partial v_k}{\partial u_j}dv^i\left(\frac{\partial}{\partial v_k}\right) \\ &= \sum_{k=1}^{n}\frac{\partial v_k}{\partial u_j}\delta_{ik} \\ &= \frac{\partial v_i}{\partial u_j}. \end{aligned}$$

This proves the transition formula for the cotangent basis:

$$dv^i = \sum_{k=1}^{n}\frac{\partial v_i}{\partial u_k}du^k. \tag{3.1}$$

Example 3.3. Consider $M = \mathbb{R}^2$ which can be parametrized by

$$\begin{aligned} \mathsf{F}_1(x,y) &= (x,y) \\ \mathsf{F}_2(r,\theta) &= (r\cos\theta, r\sin\theta). \end{aligned}$$

From (3.1), the conversion between $\{dr, d\theta\}$ and $\{dx, dy\}$ is given by:

$$\begin{aligned} dx &= \frac{\partial x}{\partial r}dr + \frac{\partial x}{\partial \theta}d\theta \\ &= (\cos\theta)\ dr - (r\sin\theta)\ d\theta \\ dy &= \frac{\partial y}{\partial r}dr + \frac{\partial y}{\partial \theta}d\theta \\ &= (\sin\theta)\ dr + (r\cos\theta)\ d\theta \end{aligned}$$

□

Exercise 3.2. Consider $M = \mathbb{R}^3$ which can be parametrized by:

$$\begin{aligned} \mathsf{F}_1(x,y,z) &= (x,y,z) \\ \mathsf{F}_2(r,\theta,z) &= (r\cos\theta, r\sin\theta, z) \\ \mathsf{F}_3(\rho,\phi,\theta) &= (\rho\sin\phi\cos\theta, \rho\sin\phi\sin\theta, \rho\cos\phi) \end{aligned}$$

Express $\{dx, dy, dz\}$ in terms of $\{dr, d\theta, dz\}$ and $\{d\rho, d\phi, d\theta\}$.

Exercise 3.3. Suppose $\mathsf{F}(u_1, \ldots, u_n)$ and $\mathsf{G}(v_1, \ldots, v_n)$ are two local parametrizations of a smooth manifold M. Let $\omega : M \to TM$ be a smooth differential 1-form such that on the overlap of local coordinates we have:

$$\omega = \sum_j a_j du^j = \sum_i b_i dv^i.$$

Find a conversion formula between a_j's and b_i's.

3.2. Tangent and Cotangent Bundles

3.2.1. Definitions. Let M be a smooth manifold. Loosely speaking, the *tangent bundle* (resp. *cotangent bundle*) are defined as the disjoint union of all tangent (resp. cotangent) spaces over the whole M. Precisely:

Definition 3.4 (Tangent and Cotangent Bundles)**.** Let M be a smooth manifold. The *tangent bundle*, denoted by TM, is defined to be:

$$TM = \bigcup_{p \in M} \left(\{p\} \times T_p M \right).$$

Elements in TM can be written as (p, V) where $V \in T_p M$.

Similarly, the *cotangent bundle*, denoted by T^*M, is defined to be:

$$T^*M = \bigcup_{p \in M} \left(\{p\} \times T_p^* M \right).$$

Elements in T^*M can be written as (p, ω) where $\omega \in T_p^* M$.

Suppose $\mathsf{F}(u_1, \ldots, u_n) : \mathcal{U} \to M$ is a local parametrization of M, then a general element of TM can be written as:

$$\left(p, \sum_{i=1}^n V^i \frac{\partial}{\partial u_i}(p) \right)$$

and a general element of T^*M can be written as:

$$\left(p, \sum_{i=1}^n a_i \, du^i \Big|_p \right).$$

We are going to explain why both TM and T^*M are smooth manifolds. The local parametrization $\mathsf{F}(u_1, \ldots, u_n)$ of M induces a local parametrization $\widetilde{\mathsf{F}}$ of TM defined by:

$$\begin{aligned} \widetilde{\mathsf{F}} &: \mathcal{U} \times \mathbb{R}^n \to TM \\ (u_1, \ldots, u_n; V^1, \ldots, V^n) &\mapsto \left(\mathsf{F}(u_1, \ldots, u_n), \sum_{i=1}^n V^i \frac{\partial}{\partial u_i}\Big|_{\mathsf{F}(u_1, \ldots, u_n)} \right). \end{aligned} \tag{3.2}$$

Likewise, it also induces a local parametrization $\widetilde{\mathsf{F}}^*$ of T^*M defined by:

$$\begin{aligned} \widetilde{\mathsf{F}}^* &: \mathcal{U} \times \mathbb{R}^n \to T^*M \\ (u_1, \ldots, u_n; a_1, \ldots, a_n) &\mapsto \left(\mathsf{F}(u_1, \ldots, u_n), \sum_{i=1}^n a_i \, du^i \Big|_{\mathsf{F}(u_1, \ldots, u_n)} \right). \end{aligned} \tag{3.3}$$

It suggests that TM and T^*M are both smooth manifolds of dimension $2 \dim M$. To do so, we need to verify compatible F's induce compatible $\widetilde{\mathsf{F}}$ and $\widetilde{\mathsf{F}}^*$. Let's state this as a proposition and we leave the proof as an exercise for readers:

Proposition 3.5. *Let M^n be a smooth manifold. Suppose F and G are two overlapping smooth local parametrizations of M, then their induced local parametrizations $\widetilde{\mathsf{F}}$ and $\widetilde{\mathsf{G}}$ defined as in* (3.2) *on the tangent bundle TM are compatible, and also that $\widetilde{\mathsf{F}}^*$ and $\widetilde{\mathsf{G}}^*$ defined as in* (3.3) *on the cotangent bundle T^*M are also compatible.*

Corollary 3.6. *The tangent bundle TM and the cotangent bundle T^*M of a smooth manifold M are both smooth manifolds of dimension $2 \dim M$.*

Exercise 3.4. Prove Proposition 3.5.

Exercise 3.5. Show that the bundle map $\pi : TM \to M$ taking $(p, V) \in TM$ to $p \in M$ is a submersion. Show also that the set:

$$\Sigma_0 := \{(p, 0) \in TM : p \in M\}$$

is a submanifold of TM.

3.2.2. Vector Fields. Intuitively, a vector field V on a manifold M is an assignment of a vector to each point on M. Therefore, it can be regarded as a map $V : M \to TM$ such that $V(p) \in \{p\} \times T_pM$. Since we have shown that the tangent bundle TM is also a smooth manifold, one can also talk about C^k and *smooth* vector fields.

Definition 3.7 (Vector Fields of Class C^k)**.** Let M be a smooth manifold. A map $V : M \to TM$ is said to be a *vector field* if for each $p \in M$, we have $V(p) = (p, V_p) \in \{p\} \times T_pM$.

If V is of class C^k as a map between M and TM, then we say V is a C^k *vector field*. If V is of class C^∞, then we say V is a *smooth vector field*.

Remark 3.8. In the above definition, we used $V(p)$ to be denote the element (p, V_p) in TM, and V_p to denote the vector in T_pM. We will distinguish between them for a short while. After getting used to the notations, we will abuse the notations and use V_p and $V(p)$ interchangeably. □

Remark 3.9. Note that a vector field can also be defined *locally* on an open set $\mathcal{O} \subset M$. In such case we say V *is a* C^k *on* $\mathcal{O}$ if the map $V : \mathcal{O} \to TM$ is C^k. □

Under a local parametrization $\mathsf{F}(u_1, \ldots, u_n) : \mathcal{U} \to M$ of M, a vector field $V : M \to TM$ can be expressed in terms of local coordinates as:

$$V(p) = \left(p, \sum_{i=1}^{n} V^i(p) \frac{\partial}{\partial u_i}(p)\right).$$

The functions $V^i : \mathsf{F}(\mathcal{U}) \subset M \to \mathbb{R}$ are all locally defined and are commonly called the *components* of V with respect to local coordinates $(u_1, \ldots, u_n)$.

Let $\widetilde{\mathsf{F}}(u_1, \ldots, u_n; V^1, \ldots, V^n)$ be the induced local parametrization of TM defined as in (3.2). Then, one can verify that:

$$\begin{aligned}\widetilde{\mathsf{F}}^{-1} \circ V \circ \mathsf{F}(u_1, \ldots, u_n) &= \widetilde{\mathsf{F}}^{-1}\left(\mathsf{F}(u_1, \ldots, u_n), \sum_{i=1}^{n} V^i(\mathsf{F}(u_1, \ldots, u_n)) \left.\frac{\partial}{\partial u_i}\right|_{\mathsf{F}(u_1, \ldots, u_n)}\right) \\ &= \Big(u_1, \ldots, u_n; V^1(\mathsf{F}(u_1, \ldots, u_n)), \ldots, V^n(\mathsf{F}(u_1, \ldots, u_n))\Big).\end{aligned}$$

Therefore, $\widetilde{\mathsf{F}}^{-1} \circ V \circ \mathsf{F}(u_1, \ldots, u_n)$ is smooth if and only if the components V^i's are all smooth. Similarly for class C^k. In short, a vector field V is smooth if and only if the components V^i in every its local expression:

$$V(p) = \left(p, \sum_{i=1}^{n} V^i(p) \frac{\partial}{\partial u_i}(p)\right)$$

are all smooth.

3.2.3. Differential 1-Forms. Differential 1-forms are the *dual* counterpart of vector fields. It is essentially an assignment of a cotangent vector to each point on M. Precisely:

Definition 3.10 (Differential 1-Forms of Class C^k)**.** Let M be a smooth manifold. A map $\omega : M \to T^*M$ is said to be a *differential 1-form* if for each $p \in M$, we have $\omega(p) = (p, \omega_p) \in \{p\} \times T_p^*M$.

If ω is of class C^k as a map between M and T^*M, then we say ω is a C^k *differential 1-form*. If ω is of class C^∞, then we say ω is a *smooth differential 1-form*.

Remark 3.11. At this moment we use $\omega(p)$ to denote an element in $\{p\} \times T_pM$, and ω_p to denote an element in T_p^*M. We will abuse the notations later on and use them interchangeably, since such a distinction is unnecessary for many practical purposes. □

Remark 3.12. If a differential 1-form ω is locally defined on an open set $\mathcal{O} \subset M$, we may say ω is C^k on $\mathcal{O}$ to mean the map $\omega : \mathcal{O} \to T^*M$ is of class C^k. □

Under a local parametrization $\mathsf{F}(u_1, \ldots, u_n) : \mathcal{U} \to M$ of M, a differential 1-form $\omega : M \to T^*M$ has a local coordinate expression given by:

$$\omega(p) = \left(p, \sum_{i=1}^{n} \omega_i(p)\, du^i\Big|_p\right)$$

where $\omega_i : \mathsf{F}(\mathcal{U}) \subset M \to \mathbb{R}$ are locally defined functions and are commonly called the *components* of ω with respect to local coordinates $(u_1, \ldots, u_n)$. Similarly to vector fields, one can show that ω is a C^∞ differential 1-form if and only if all ω_i's are smooth under any local coordinates in the atlas of M (see Exercise 3.6).

Exercise 3.6. Show that a differential 1-form ω is C^k on M if and only if all components ω_i's are C^k under any local coordinates in the atlas of M.

Example 3.13. The differential 1-form:

$$\omega = -\frac{y}{x^2 + y^2}\, dx + \frac{x}{x^2 + y^2}\, dy$$

is smooth on $\mathbb{R}^2 \backslash \{0\}$, but is not smooth on $\mathbb{R}^2$. □

3.2.4. Push-Forward and Pull-Back. Consider a smooth map $\Phi : M \to N$ between two smooth manifolds M and N. The tangent map at p denoted by $(\Phi_*)_p$ is the induced map between tangent spaces T_pM and $T_{\Phi(p)}N$. Apart from calling it the tangent map, we often call Φ_* to be the *push-forward by* Φ, since Φ and Φ_* are both from the space M to the space N.

The push-forward map Φ_* takes tangent vectors on M to tangent vectors on N. There is another induced map Φ^*, called the *pull-back by* Φ, which is *loosely* defined as follows:

$$(\Phi^*\omega)(V) = \omega(\Phi_* V)$$

where ω is a cotangent vector and V is a tangent vector. In order for the above to make sense, V has to be a tangent vector on M (say at p). Then, $\Phi_* V$ is a tangent vector in $T_{\Phi(p)}N$. Therefore, $\Phi^*\omega$ needs to act on V and hence is a cotangent vector in T_p^*M; whereas ω acts on $\Phi_* V$ and so it should be a cotangent vector in $T^*_{\Phi(p)}N$. It is precisely defined as follows:

Definition 3.14 (Pull-Back of Cotangent Vectors)**.** Let $\Phi : M \to N$ be a smooth map between two smooth manifolds M and N. Given any cotangent vector $\omega_{\Phi(p)} \in T^*_{\Phi(p)}N$, the *pull-back of* ω *by* Φ *at* p denoted by $(\Phi^*\omega)_p$ is an element in T^*_pM and is defined to be the following linear functional on T_pM:

$$(\Phi^*\omega)_p : T_pM \to \mathbb{R}$$
$$(\Phi^*\omega)_p (V_p) := \omega_{\Phi(p)} \big((\Phi_*)_p(V_p)\big)$$

Therefore, one can think of Φ^* is a map which takes a *cotangent vector* $\omega_{\Phi(p)} \in T^*_{\Phi(p)}N$ to a cotangent vector $(\Phi^*\omega)_p$ on T^*_pM. As it is in the opposite direction to $\Phi : M \to N$, we call Φ^* the *pull-back* whereas Φ_* is called the *push-forward*.

Remark 3.15. In many situations, the points p and $\Phi(p)$ are clear from the context. Therefore, we often omit the subscripts p and $\Phi(p)$ when dealing with pull-backs and push-forwards. □

Example 3.16. Consider the map $\Phi : \mathbb{R} \to \mathbb{R}^2\backslash\{0\}$ defined by:

$$\Phi(\theta) = (\cos\theta, \sin\theta).$$

Let ω be the following 1-form on $\mathbb{R}^2\backslash\{0\}$:

$$\omega = -\frac{y}{x^2+y^2}\,dx + \frac{x}{x^2+y^2}\,dy.$$

First note that

$$\Phi_*\left(\frac{\partial}{\partial\theta}\right) = \frac{\partial\Phi}{\partial\theta} = \frac{\partial\overbrace{(\cos\theta)}^{x}}{\partial\theta}\frac{\partial}{\partial x} + \frac{\partial\overbrace{(\sin\theta)}^{y}}{\partial\theta}\frac{\partial}{\partial y} = -y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y}.$$

Therefore, one can compute:

$$\begin{aligned}(\Phi^*\omega)\left(\frac{\partial}{\partial\theta}\right) &= \omega\left(\Phi_*\left(\frac{\partial}{\partial\theta}\right)\right) = \omega\left(-y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y}\right) \\ &= -y\left(\frac{-y}{x^2+y^2}\right) + x\left(\frac{x}{x^2+y^2}\right) \\ &= 1.\end{aligned}$$

Therefore, $\Phi^*\omega = d\theta$. □

Example 3.17. Let $M := \mathbb{R}^2\backslash\{(0,0)\}$ (equipped with polar (r,θ)-coordinates) and $N = \mathbb{R}^2$ (with (x,y)-coordinates), and define:

$$\Phi : M \to N$$
$$\Phi(r,\theta) := (r\cos\theta, r\sin\theta)$$

One can verify that:

$$\begin{aligned}\Phi_*\left(\frac{\partial}{\partial r}\right) &= \frac{\partial\Phi}{\partial r} = (\cos\theta)\frac{\partial}{\partial x} + (\sin\theta)\frac{\partial}{\partial y} \\ \Phi_*\left(\frac{\partial}{\partial\theta}\right) &= \frac{\partial\Phi}{\partial\theta} = (-r\sin\theta)\frac{\partial}{\partial x} + (r\cos\theta)\frac{\partial}{\partial y} \\ &= -y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y}\end{aligned}$$

Hence, we have:

$$\begin{aligned}
(\Phi^* dx)\left(\frac{\partial}{\partial r}\right) &= dx\left(\Phi_*\left(\frac{\partial}{\partial r}\right)\right) \\
&= dx\left((\cos\theta)\frac{\partial}{\partial x} + (\sin\theta)\frac{\partial}{\partial y}\right) \\
&= \cos\theta \\
(\Phi^* dx)\left(\frac{\partial}{\partial \theta}\right) &= dx\left(\Phi_*\left(\frac{\partial}{\partial \theta}\right)\right) \\
&= dx\left(-y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y}\right) \\
&= -y = -r\sin\theta
\end{aligned}$$

We conclude:

$$\Phi^* dx = \cos\theta\, dr - r\sin\theta\, d\theta.$$

□

Given a smooth map $\Phi : M^m \to N^n$, and local coordinates $(u_1, \ldots, u_m)$ of M around p and local coordinates $(v_1, \ldots, v_n)$ of N around $\Phi(p)$. One can compute a local expression for Φ^*:

$$\Phi^* dv^i = \sum_{j=1}^{n} \frac{\partial v_i}{\partial u_j} du^j \tag{3.4}$$

where $(v_1, \ldots, v_n)$ is regarded as a function of $(u_1, \ldots, u_m)$ via the map $\Phi : M \to N$.

Exercise 3.7. Prove (3.4).

Exercise 3.8. Express $\Phi^* dy$ in terms of dr and $d\theta$ in Example 3.17. Try computing it directly and then verify that (3.4) gives the same result.

Exercise 3.9. Denote (x_1, x_2) the coordinates for $\mathbb{R}^2$ and (y_1, y_2, y_3) the coordinates for $\mathbb{R}^3$. Define the map $\Phi : \mathbb{R}^2 \to \mathbb{R}^3$ by:

$$\Phi(x_1, x_2) = (x_1 x_2, x_2 x_3, x_3 x_1).$$

Compute $\Phi^*(dy^1)$, $\Phi^*(dy^2)$ and $\Phi^*(dy^3)$.

Exercise 3.10. Consider the map $\Phi : \mathbb{R}^3 \backslash \{0\} \to \mathbb{RP}^2$ defined by:

$$\Phi(x, y, z) = [x : y : z].$$

Consider the local parametrization $\mathsf{F}(u_1, u_2) = [1 : u_1 : u_2]$ of $\mathbb{RP}^2$. Compute $\Phi^*(du^1)$ and $\Phi^*(du^2)$.

3.3. Tensor Products

In Differential Geometry, tensor products are often used to produce bilinear, or in general multilinear, maps between tangent and cotangent spaces. The first and second fundamental forms of a regular surface, the Riemann curvature, etc. can all be expressed using tensor notations.

3.3.1. Tensor Products in Vector Spaces. Given two vector spaces V and W, their dual spaces V^* and W^* are vector spaces of all linear functionals $T : V \to \mathbb{R}$ and $S : W \to \mathbb{R}$ respectively. Pick two linear functionals $T \in V^*$ and $S \in W^*$, their tensor product $T \otimes S$ is a map from $V \times W$ to $\mathbb{R}$ defined by:

$$\begin{aligned} T \otimes S &: V \times W \to \mathbb{R} \\ (T \otimes S)(X, Y) &:= T(X)\, S(Y) \end{aligned}$$

It is easy to verify that $T \otimes S$ is bilinear, meaning that it is linear at each slot:

$$\begin{aligned} &(T \otimes S)\,(a_1 X_1 + a_2 X_2,\, b_1 Y_1 + b_2 Y_2) \\ &= a_1 b_1 (T \otimes S)(X_1, Y_1) + a_2 b_1 (T \otimes S)(X_2, Y_1) \\ &\quad + a_1 b_2 (T \otimes S)(X_1, Y_2) + a_2 b_2 (T \otimes S)(X_1, Y_2) \end{aligned}$$

Given three vector spaces U, V, W, and linear functionals $T_U \in U^*$, $T_V \in V^*$ and $T_W \in W^*$, one can define a triple tensor product $T_U \otimes (T_V \otimes T_W)$ by:

$$\begin{aligned} T_U \otimes (T_V \otimes T_W) &: U \times (V \times W) \to \mathbb{R} \\ (T_U \otimes (T_V \otimes T_W))(X, Y, Z) &:= T_U(X)\, (T_V \otimes T_W)(Y, Z) \\ &= T_U(X)\, T_V(Y)\, T_W(Z) \end{aligned}$$

One check easily that $(T_U \otimes T_V) \otimes T_W = T_U \otimes (T_V \otimes T_W)$. Since there is no ambiguity, we may simply write $T_U \otimes T_V \otimes T_W$. Inductively, given finitely many vector spaces $V_1, \ldots, V_k$, and linear functions $T_i \in V_i^*$, we can define the tensor product $T_1 \otimes \cdots \otimes T_k$ as a k-linear map by:

$$\begin{aligned} T_1 \otimes \cdots \otimes T_k &: V_1 \times \cdots \times V_k \to \mathbb{R} \\ (T_1 \otimes \cdots \otimes T_k)(X_1, \ldots, X_k) &:= T_1(X_1) \cdots T_k(X_k) \end{aligned}$$

Given two tensor products $T_1 \otimes S_1 : V \times W \to \mathbb{R}$ and $T_2 \otimes S_2 : V \times W \to \mathbb{R}$, one can form a linear combination of them:

$$\begin{aligned} \alpha_1 (T_1 \otimes S_1) + \alpha_2 (T_2 \otimes S_2) &: V \times W \to \mathbb{R} \\ (\alpha_1 (T_1 \otimes S_1) + \alpha_2 (T_2 \otimes S_2))(X, Y) &:= \alpha_1 (T_1 \otimes S_1)(X, Y) + \alpha_2 (T_2 \otimes S_2)(X, Y) \end{aligned}$$

The tensor products $T \otimes S$ with $T \in V^*$ and $S \in W^*$ generate a vector space. We denote this vector space by:

$$V^* \otimes W^* := \operatorname{span}\{T \otimes S : T \in V^* \text{ and } S \in W^*\}.$$

Exercise 3.11. Verify that $\alpha\,(T \otimes S) = (\alpha T) \otimes S = T \otimes (\alpha S)$. Therefore, we can simply write $\alpha T \otimes S$.

Exercise 3.12. Show that the tensor product is bilinear in a sense that:

$$T \otimes (\alpha_1 S_1 + \alpha_2 S_2) = \alpha_1 T \otimes S_1 + \alpha_2 T \otimes S_2$$

and similar for the T slot.

Let's take the dual basis as an example to showcase the use of tensor products. Consider a vector space V with a basis $\{\mathsf{e}_i\}_{i=1}^n$. Let $\{\mathsf{e}_i^*\}_{i=1}^n$ be its dual basis for V^*. Then, one can check that:

$$\begin{aligned}(\mathsf{e}_i^* \otimes \mathsf{e}_j^*)(\mathsf{e}_k, \mathsf{e}_l) &= \mathsf{e}_i^*(\mathsf{e}_k)\, \mathsf{e}_k^*(\mathsf{e}_l) \\ &= \delta_{ik}\,\delta_{jl} \\ &= \begin{cases} 1 & \text{if } i = k \text{ and } j = l \\ 0 & \text{otherwise} \end{cases}\end{aligned}$$

Generally, the sum $\sum_{i,j=1}^{n} A_{ij}\mathsf{e}_i^* \otimes \mathsf{e}_j^*$ will act on vectors in V by:

$$\begin{aligned}&\left(\sum_{i,j=1}^{n} A_{ij}\mathsf{e}_i^* \otimes \mathsf{e}_j^*\right)\left(\sum_{k=1}^{n} \alpha_k \mathsf{e}_k, \sum_{l=1}^{n} \beta_l \mathsf{e}_l\right) \\ &= \sum_{i,j,k,l=1}^{n} A_{ij}\alpha_k\beta_l (\mathsf{e}_i^* \otimes \mathsf{e}_j^*)(\mathsf{e}_k, \mathsf{e}_l) = \sum_{i,j,k,l=1}^{n} A_{ij}\alpha_k\beta_l\delta_{ik}\delta_{jl} = \sum_{k,l=1}^{n} A_{kl}\alpha_k\beta_l\end{aligned}$$

In other words, the sum of tensor products $\sum_{i,j=1}^{n} A_{ij}\mathsf{e}_i^* \otimes \mathsf{e}_j^*$ is the inner product on V represented by the matrix $[A_{kl}]$ with respect to the basis $\{\mathsf{e}_i\}_{i=1}^n$ of V. For example, when $A_{kl} = \delta_{kl}$, then $\sum_{i,j=1}^{n} A_{ij}\mathsf{e}_i^* \otimes \mathsf{e}_j^* = \sum_{i=1}^{n} \mathsf{e}_i^* \otimes \mathsf{e}_i^*$. It is the usual dot product on V.

Exercise 3.13. Show that $\{\mathsf{e}_i^* \otimes \mathsf{e}_j^*\}_{i,j=1}^n$ is a basis for $V^* \otimes V^*$. What is the dimension of $V^* \otimes V^*$?

Exercise 3.14. Suppose $\dim V = 2$. Let $\omega \in V^* \otimes V^*$ satisfy:

$$\begin{aligned}\omega(\mathsf{e}_1, \mathsf{e}_1) &= 0 & \omega(\mathsf{e}_1, \mathsf{e}_2) &= 3 \\ \omega(\mathsf{e}_2, \mathsf{e}_1) &= -3 & \omega(\mathsf{e}_2, \mathsf{e}_2) &= 0\end{aligned}$$

Express ω in terms of e_i^*'s.

To describe linear or multilinear map between two vector spaces V and W (where W is not necessarily the one-dimensional space $\mathbb{R}$), one can also use tensor products. Given a linear functional $f \in V^*$ and a vector $w \in W$, we can form a tensor $f \otimes w$, which is regarded as a linear map $f \otimes w : V \to W$ defined by:

$$(f \otimes w)(v) := f(v)w.$$

Let $\{\mathsf{e}_i\}$ be a basis for V, and $\{\mathsf{f}_j\}$ be a basis for W. Any linear map $T : V \to W$ can be expressed in terms of these bases. Suppose:

$$T(\mathsf{e}_i) = \sum_j A_i^j \mathsf{f}_j.$$

Then, we claim that T can be expressed using the following tensor notations:

$$T = \sum_{i,j} A_i^j \mathsf{e}_i^* \otimes \mathsf{f}_j$$

Let's verify this. Note that a linear map is determined by its action on the basis $\{\mathsf{e}_i\}$ for V. It suffices to show:

$$\left(\sum_{i,j} A_i^j \mathsf{e}_i^* \otimes \mathsf{f}_j\right)(\mathsf{e}_k) = T(\mathsf{e}_k).$$

Using the fact that:

$$(\mathsf{e}_i^* \otimes \mathsf{f}_j)(\mathsf{e}_k) = \mathsf{e}_i^*(\mathsf{e}_k)\mathsf{f}_j = \delta_{ik}\mathsf{f}_j,$$

one can compute:

$$\left(\sum_{i,j} A_i^j \mathsf{e}_i^* \otimes \mathsf{f}_j\right)(\mathsf{e}_k) = \sum_{i,j} A_i^j (\mathsf{e}_i^* \otimes \mathsf{f}_j)(\mathsf{e}_k)$$
$$= \sum_{i,j} A_i^j \delta_{ik} \mathsf{f}_j = \sum_j A_k^j \mathsf{f}_j = T(\mathsf{e}_k)$$

as desired.

Generally, if $T_1, \ldots, T_k \in V^*$ and $X \in V$, then

$$T_1 \otimes \cdots \otimes T_k \otimes X$$

is regarded to be a k-linear map from $V \times \ldots \times V$ to V, defined by:

$$T_1 \otimes \cdots \otimes T_k \otimes X : \underbrace{V \times \ldots \times V}_{k} \to V$$
$$(T_1 \otimes \cdots \otimes T_k \otimes X)(Y_1, \ldots, Y_k) := T_1(Y_1) \cdots T_k(Y_k)\, X$$

Example 3.18. One can write the cross-product in $\mathbb{R}^3$ using tensor notations. Think of the cross product as a bilinear map $\omega : \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}^3$ that takes two input vectors u and v, and outputs the vector $\mathsf{u} \times \mathsf{v}$. Let $\{\mathsf{e}_1, \mathsf{e}_2, \mathsf{e}_3\}$ be the standard basis in $\mathbb{R}^3$ (i.e. $\{\mathsf{i}, \mathsf{j}, \mathsf{k}\}$). Then one can write:

$$\begin{aligned} \omega = {} & \mathsf{e}_1^* \otimes \mathsf{e}_2^* \otimes \mathsf{e}_3 - \mathsf{e}_2^* \otimes \mathsf{e}_1^* \otimes \mathsf{e}_3 \\ & + \mathsf{e}_2^* \otimes \mathsf{e}_3^* \otimes \mathsf{e}_1 - \mathsf{e}_3^* \otimes \mathsf{e}_2^* \otimes \mathsf{e}_1 \\ & + \mathsf{e}_3^* \otimes \mathsf{e}_1^* \otimes \mathsf{e}_2 - \mathsf{e}_1^* \otimes \mathsf{e}_3^* \otimes \mathsf{e}_2 \end{aligned}$$

One can check that, for instance, $\omega(\mathsf{e}_1, \mathsf{e}_2) = \mathsf{e}_3$, which is exactly $\mathsf{e}_1 \times \mathsf{e}_2 = \mathsf{e}_3$. □

3.3.2. Tensor Products on Smooth Manifolds. In the previous subsection we take tensor products on a general abstract vector space V. In this course, we will mostly deal with the case when V is the tangent or cotangent space of a smooth manifold M.

Recall that if $\mathsf{F}(u_1, \ldots, u_n)$ is a local parametrization of M, then there is a local coordinate basis $\left\{\frac{\partial}{\partial u_i}(p)\right\}_{j=1}^n$ for the tangent space T_pM at every $p \in M$ covered by F. The cotangent space T_p^*M has a dual basis $\left\{du^j\big|_p\right\}_{j=1}^n$ defined by $du_j\left(\frac{\partial}{\partial u_i}\right) = \delta_{ij}$ at every $p \in M$.

Then, one can take tensor products of du^i's and $\frac{\partial}{\partial u_i}$'s to express multilinear maps between tangent and cotangent spaces. For instance, the tensor product $g = \sum_{i,j=1}^n g_{ij} du^i \otimes du^j$, where g_{ij}'s are scalar functions, means that it is a bilinear map at each point $p \in M$ such that:

$$g(X, Y) = \sum_{i,j=1}^n g_{ij}(du^i \otimes du^j)(X, Y) = \sum_{i,j=1}^n g_{ij} du^i(X)\, du^j(Y)$$

for any vector field $X, Y \in TM$. In particular, we have:

$$g\left(\frac{\partial}{\partial u_i}, \frac{\partial}{\partial u_j}\right) = g_{ij}.$$

We can also express multilinear maps from $T_pM \times T_pM \times T_pM$ to T_pM. For instance, we let:

$$\mathrm{Rm} = \sum_{i,j,k,l=1}^{n} R_{ijk}^{l} du^i \otimes du^j \otimes du^k \otimes \frac{\partial}{\partial u_l}.$$

Then, Rm is a mutlilinear map at each $p \in M$ such that:

$$\mathrm{Rm}(X, Y, Z) = \sum_{i,j,k,l=1}^{n} R_{ijk}^{l} du^i(X)\, du^j(Y)\, du^k(Z)\, \frac{\partial}{\partial u_l}.$$

It is a trilinear map such that:

$$\mathrm{Rm}\left(\frac{\partial}{\partial u_i}, \frac{\partial}{\partial u_j}, \frac{\partial}{\partial u_k}\right) = \sum_{l=1}^{n} R_{ijk}^{l} \frac{\partial}{\partial u_l}.$$

We call g a $(2,0)$-tensor (meaning that it maps two vectors to a scalar), and Rm a $(3,1)$-tensor (meaning that it maps three vectors to one vector). In general, we can also define $(k,0)$-tensor ω on M which has the general form:

$$\omega_p = \sum_{i_1,\ldots,i_k=1}^{n} \omega_{i_1 i_2 \cdots i_k}(p)\, du^{i_1}\Big|_p \otimes \cdots \otimes du^{i_k}\Big|_p$$

Here $\omega_{i_1 i_2 \cdots i_k}$'s are scalar functions. This tensor maps the tangent vectors $\left(\frac{\partial}{\partial u_{i_1}}, \ldots, \frac{\partial}{\partial u_{i_k}}\right)$ to the scalar $\omega_{i_1 i_2 \ldots i_k}$ at the corresponding point.

Like the Rm-tensor, we can also generally define $(k,1)$-tensor Ω on M which has the general form:

$$\Omega_p = \sum_{i_1,\ldots,i_k,j=1}^{n} \Omega_{i_1 i_2 \cdots i_k}^{j}(p)\, du^{i_1}\Big|_p \otimes \cdots \otimes du^{i_k}\Big|_p \otimes \frac{\partial}{\partial u_j}(p)$$

where $\Omega_{i_1 i_2 \ldots i_k}^{j}$'s are scalar functions. This tensor maps the tangent vectors $\left(\frac{\partial}{\partial u_{i_1}}, \ldots, \frac{\partial}{\partial u_{i_k}}\right)$ to the tangent vector $\sum_j \Omega_{i_1 i_2 \ldots i_k}^{j} \frac{\partial}{\partial u_j}$ at the corresponding point.

Note that these g_{ij}, R_{ijk}^{l}, $\omega_{i_1 i_2 \cdots i_k}$ and $\Omega_{i_1 i_2 \ldots i_k}^{j}$ are scalar functions locally defined on the open set covered by the local parametrization F, so we can talk about whether they are smooth or not:

Definition 3.19 (Smooth Tensors on Manifolds)**.** A *smooth* $(k,0)$*-tensor* ω on M is a k-linear map $\omega_p : \underbrace{T_pM \times \ldots \times T_pM}_{k} \to \mathbb{R}$ at each $p \in M$ such that under any local parametrization $\mathsf{F}(u_1, \ldots, u_n) : \mathcal{U} \to M$, it can be written in the form:

$$\omega_p = \sum_{i_1,\ldots,i_k=1}^{n} \omega_{i_1 i_2 \cdots i_k}(p)\, du^{i_1}\Big|_p \otimes \cdots \otimes du^{i_k}\Big|_p$$

where $\omega_{i_1 i_2 \ldots i_k}$'s are smooth scalar functions locally defined on $\mathsf{F}(\mathcal{U})$.

A *smooth* $(k,1)$*-tensor* Ω on M is a k-linear map $\Omega_p : \underbrace{T_pM \times \ldots \times T_pM}_{k} \to T_pM$ at each $p \in M$ such that under any local parametrization $\mathsf{F}(u_1, \ldots, u_n) : \mathcal{U} \to M$, it can be written in the form:

$$\Omega_p = \sum_{i_1,\ldots,i_k,j=1}^{n} \Omega^j_{i_1 i_2 \cdots i_k}(p)\, du^{i_1}\Big|_p \otimes \cdots \otimes du^{i_k}\Big|_p \otimes \frac{\partial}{\partial u_j}(p)$$

where $\Omega^j_{i_1 i_2 \ldots i_k}$'s are smooth scalar functions locally defined on $\mathsf{F}(\mathcal{U})$.

Remark 3.20. Since T_pM is finite dimensional, from Linear Algebra we know $(T_pM)^{**}$ is isomorphic to T_pM. Therefore, a tangent vector $\frac{\partial}{\partial u_i}(p)$ can be regarded as a linear functional on cotangent vectors in T_p^*M, meaning that:

$$\frac{\partial}{\partial u_i}\bigg|_p \left(du^j\Big|_p \right) = \delta_{ij}.$$

Under this interpretation, one can also view a $(k,1)$-tensor Ω as a $(k+1)$-linear map $\Omega_p : \underbrace{T_pM \times \ldots \times T_pM}_{k} \times T_p^*M \to \mathbb{R}$, which maps $\left(du^{i_1}, \ldots, du^{i_k}, \frac{\partial}{\partial u_j}\right)$ to $\Omega^j_{i_1 i_2 \ldots i_k}$. However, we will not view a $(k,1)$-tensor this way in this course.

Generally, we can also talk about (k,s)-tensors, which is a $(k+s)$-linear map $\Omega_p : \underbrace{T_pM \times \ldots \times T_pM}_{k} \times \underbrace{T_p^*M \times \ldots \times T_p^*M}_{s} \to \mathbb{R}$ taking $\left(du^{i_1}, \ldots, du^{i_k}, \frac{\partial}{\partial u_{j_1}}, \ldots, \frac{\partial}{\partial u_{j_s}}\right)$ to a scalar. However, we seldom deal with these tensors in this course. □

Exercise 3.15. Let M be a smooth manifold with local coordinates (u_1, u_2). Consider the tensor products:

$$T_1 = du^1 \otimes du^2 \quad \text{and} \quad T_2 = du^1 \otimes \frac{\partial}{\partial u_2}.$$

Which of the following is well-defined?

(a) $T_1\left(\frac{\partial}{\partial u_1}\right)$

(b) $T_2\left(\frac{\partial}{\partial u_1}\right)$

(c) $T_1\left(\frac{\partial}{\partial u_1}, \frac{\partial}{\partial u_2}\right)$

(d) $T_2\left(\frac{\partial}{\partial u_1}, \frac{\partial}{\partial u_2}\right)$

Exercise 3.16. let M be a smooth manifold with local coordinates (u_1, u_2). The linear map $T : T_pM \to T_pM$ satisfies:

$$T\left(\frac{\partial}{\partial u_1}\right) = \frac{\partial}{\partial u_1} + \frac{\partial}{\partial u_2}$$
$$T\left(\frac{\partial}{\partial u_2}\right) = \frac{\partial}{\partial u_1} - \frac{\partial}{\partial u_2}.$$

Express T using tensor products.

One advantage of using tensor notations, instead of using matrices, to denote a multilinear map between tangent or cotangent spaces is that one can figure out the conversion rule between local coordinate systems easily (when compared to using matrices)

Example 3.21. Consider the extended complex plane $M := \mathbb{C} \cup \{\infty\}$ defined in Example 2.12. We cover M by two local parametrizations:

$$\begin{aligned} \mathsf{F}_1 : \mathbb{R}^2 &\to \mathbb{C} \subset M & \qquad \mathsf{F}_2 : \mathbb{R}^2 &\to (\mathbb{C}\backslash\{0\}) \cup \{\infty\} \subset M \\ (x, y) &\mapsto x + yi & (u, v) &\mapsto \frac{1}{u + vi} \end{aligned}$$

The transition maps on the overlap are given by:

$$(u, v) = \mathsf{F}_2^{-1} \circ \mathsf{F}_1(x, y) = \left(\frac{x}{x^2 + y^2}, -\frac{y}{x^2 + y^2}\right)$$
$$(x, y) = \mathsf{F}_1^{-1} \circ \mathsf{F}_2(u, v) = \left(\frac{u}{u^2 + v^2}, -\frac{v}{u^2 + v^2}\right)$$

Consider the $(2, 0)$-tensor ω defined using local coordinates (x, y) by:

$$\omega = e^{-(x^2+y^2)}\, dx \otimes dy.$$

Using the chain rule, we can express dx and dy in terms of du and dv:

$$\begin{aligned} dx &= d\left(\frac{u}{u^2 + v^2}\right) = \frac{(u^2 + v^2)\, du - u(2u\, du + 2v\, dv)}{(u^2 + v^2)^2} \\ &= \frac{v^2 - u^2}{(u^2 + v^2)^2} du - \frac{2uv}{(u^2 + v^2)^2} dv \\ dy &= -d\left(\frac{v}{u^2 + v^2}\right) = -\frac{(u^2 + v^2)dv - v(2u\, du + 2v\, dv)}{(u^2 + v^2)^2} \\ &= -\frac{2uv}{(u^2 + v^2)^2} du + \frac{v^2 - u^2}{(u^2 + v^2)^2} dv \end{aligned}$$

Therefore, we get:

$$\begin{aligned} dx \otimes dy = {} & \frac{2uv(u^2 - v^2)}{(u^2 + v^2)^4} du \otimes du + \frac{(u^2 - v^2)^2}{(u^2 + v^2)^4} du \otimes dv \\ & + \frac{4u^2v^2}{(u^2 + v^2)^4} dv \otimes du + \frac{2uv(u^2 - v^2)}{(u^2 + v^2)^4} dv \otimes dv \end{aligned}$$

Recall that $\omega = e^{-(x^2+y^2)}\, dx \otimes dy$, and in terms of (u, v), we have:

$$e^{-(x^2+y^2)} = e^{-\frac{1}{u^2+v^2}}.$$

Hence, in terms of (u, v), ω is expressed as:

$$\omega = e^{-\frac{1}{u^2+v^2}} \left\{ \frac{2uv(u^2 - v^2)}{(u^2 + v^2)^4} du \otimes du + \frac{(u^2 - v^2)^2}{(u^2 + v^2)^4} du \otimes dv \right.$$
$$\left. + \frac{4u^2v^2}{(u^2 + v^2)^4} dv \otimes du + \frac{2uv(u^2 - v^2)}{(u^2 + v^2)^4} dv \otimes dv \right\}$$

□

Exercise 3.17. Consider the extended complex plane $\mathbb{C} \cup \{\infty\}$ as in Example 3.21, and the $(1, 1)$-tensor of the form:

$$\Omega = e^{-(x^2+y^2)}\, dx \otimes \frac{\partial}{\partial y}.$$

Express Ω in terms of (u, v).

Generally, if $(u_1, \ldots, u_n)$ and $(v_1, \ldots, v_n)$ are two overlapping local coordinates on a smooth manifold M, then given a $(2, 0)$-tensor:

$$g = \sum_{i,j} g_{ij} du^i \otimes du^j$$

written using the u_i's coordinates, one can convert it to v_α's coordinates by the chain rule:

$$g = \sum_{i,j} g_{ij} du^i \otimes du^j = \sum_{i,j} g_{ij} \left(\sum_\alpha \frac{\partial u_i}{\partial v_\alpha} dv^\alpha \right) \otimes \left(\sum_\beta \frac{\partial u_j}{\partial v_\beta} dv^\beta \right)$$
$$= \sum_{\alpha,\beta} \left(\sum_{i,j} g_{ij} \frac{\partial u_i}{\partial v_\alpha} \frac{\partial u_j}{\partial v_\beta} \right) dv^\alpha \otimes dv^\beta$$

Exercise 3.18. Given that u_i's and v_α's are overlapping local coordinates of a smooth manifold M. Using these coordinates, one can express the following $(3, 1)$-tensor in two ways:

$$\mathrm{Rm} = \sum_{i,j,k,l} R^l_{ijk} du^i \otimes du^j \otimes du^k \otimes \frac{\partial}{\partial u_l} = \sum_{\alpha,\beta,\gamma,\eta} \widetilde{R}^\eta_{\alpha\beta\gamma} dv^\alpha \otimes dv^\beta \otimes dv^\gamma \otimes \frac{\partial}{\partial v_\eta}$$

Express R^l_{ijk} in terms of $R^\eta_{\alpha\beta\gamma}$'s.

Exercise 3.19. Given that u_i's and v_α's are overlapping local coordinates of a smooth manifold M. Suppose g and h are two $(2, 0)$-tensors expressed in terms of local coordinates as:

$$g = \sum_{i,j} g_{ij}\, du^i \otimes du^j = \sum_{\alpha,\beta} \widetilde{g}_{\alpha\beta} dv^\alpha \otimes dv^\beta$$
$$h = \sum_{i,j} h_{ij}\, du^i \otimes du^j = \sum_{\alpha,\beta} \widetilde{h}_{\alpha\beta} dv^\alpha \otimes dv^\beta.$$

Let G be the matrix with g_{ij} as its (i, j)-th entry, and let g^{ij} be the (i, j)-th entry of G^{-1}. Similarly, define $\widetilde{g}^{\alpha\beta}$ to be the inverse of $\widetilde{g}_{\alpha\beta}$. Show that:

$$\sum_{i,j} g^{ik} h_{kj}\, du^i \otimes du^j = \sum_{\alpha,\beta} \widetilde{g}^{\alpha\gamma} \widetilde{h}_{\gamma\beta}\, dv^\alpha \otimes dv^\beta.$$

3.4. Wedge Products

Recall that in Multivariable Calculus, the cross product plays a crucial role in many aspects. It is a bilinear map which takes two vectors to one vectors, and so it is a $(2,1)$-tensor on $\mathbb{R}^3$.

Also, there is no doubt that determinant is another important quantity in Multivariable Calculus and Linear Algebra. Using tensor languages, an $n \times n$ determinant can be regarded as a n-linear map taking n vectors in $\mathbb{R}^n$ to a scalar. For instance, for the 2×2 case, one can view:

$$\det \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

as a bilinear map taking vectors (a,b) and (c,d) in $\mathbb{R}^2$ to a number $ad - bc$. Therefore, it is a $(2,0)$-tensor on $\mathbb{R}^2$; and generally for $n \times n$, the determinant is an $(n,0)$-tensor on $\mathbb{R}^n$.

Both the cross product in $\mathbb{R}^3$ and determinant ($n \times n$ in general) are *alternating*, in a sense that interchanging any pair of inputs will give a negative sign for the output. For the cross product, we have $\mathsf{a} \times \mathsf{b} = -\mathsf{b} \times \mathsf{a}$; and for the determinant, switching any pair of rows will give a negative sign:

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = - \begin{vmatrix} b & a \\ d & c \end{vmatrix}.$$

In the previous section we have seen how to express k-linear maps over tangent vectors using tensor notations. To deal with the above alternating tensors, it is more elegant and concise to use *alternating tensors*, or *wedge products* that we are going to learn in this section.

3.4.1. Wedge Product on Vector Spaces. Let's start from the easiest case. Suppose V is a finite dimensional vector space and V^* is the dual space of V. Given any two elements $T, S \in V^*$, the tensor product $T \otimes S$ is a map given by:

$$(T \otimes S)(X,Y) = T(X)\,S(Y)$$

for any $X, Y \in V$. The *wedge product* $T \wedge S$, where $T, S \in V^*$, is a bilinear map defined by:

$$T \wedge S := T \otimes S - S \otimes T$$

meaning that for any $X, Y \in V$, we have:

$$\begin{aligned}(T \wedge S)(X,Y) &= (T \otimes S)(X,Y) - (S \otimes T)(X,Y) \\ &= T(X)\,S(Y) - S(X)\,T(Y)\end{aligned}$$

It is easy to note that $T \wedge S = -S \wedge T$.

Take the cross product in $\mathbb{R}^3$ as an example. Write the cross product as a bilinear map $\omega(\mathsf{a},\mathsf{b}) := \mathsf{a} \times \mathsf{b}$. It is a $(2,1)$-tensor on $\mathbb{R}^3$ which can be represented as:

$$\begin{aligned}\omega = {}& \mathsf{e}_1^* \otimes \mathsf{e}_2^* \otimes \mathsf{e}_3 - \mathsf{e}_2^* \otimes \mathsf{e}_1^* \otimes \mathsf{e}_3 \\ &+ \mathsf{e}_2^* \otimes \mathsf{e}_3^* \otimes \mathsf{e}_1 - \mathsf{e}_3^* \otimes \mathsf{e}_2^* \otimes \mathsf{e}_1 \\ &+ \mathsf{e}_3^* \otimes \mathsf{e}_1^* \otimes \mathsf{e}_2 - \mathsf{e}_1^* \otimes \mathsf{e}_3^* \otimes \mathsf{e}_2\end{aligned}$$

Now using the wedge product notations, we can express ω as:

$$\omega = (\mathsf{e}_1^* \wedge \mathsf{e}_2^*) \otimes \mathsf{e}_3 + (\mathsf{e}_2^* \wedge \mathsf{e}_3^*) \otimes \mathsf{e}_1 + (\mathsf{e}_3^* \wedge \mathsf{e}_1^*) \otimes \mathsf{e}_2$$

which is a half shorter than using tensor products alone.

Now given three elements $T_1, T_2, T_3 \in V^*$, one can also form a triple wedge product $T_1 \wedge T_2 \wedge T_3$ which is a $(3,0)$-tensor so that switching any pair of T_i and T_j (with $i \neq j$) will give a negative sign. For instance:

$$T_1 \wedge T_2 \wedge T_3 = -T_2 \wedge T_1 \wedge T_3 \quad \text{and} \quad T_1 \wedge T_2 \wedge T_3 = -T_3 \wedge T_2 \wedge T_1.$$

It can be defined in a precise way as:

$$\begin{aligned} T_1 \wedge T_2 \wedge T_3 := & \; T_1 \otimes T_2 \otimes T_3 - T_1 \otimes T_3 \otimes T_2 \\ & + T_2 \otimes T_3 \otimes T_1 - T_2 \otimes T_1 \otimes T_3 \\ & + T_3 \otimes T_1 \otimes T_2 - T_3 \otimes T_2 \otimes T_1 \end{aligned}$$

Exercise 3.20. Verify that the above definition of triple wedge product will result in $T_1 \wedge T_2 \wedge T_3 = -T_3 \wedge T_2 \wedge T_1$.

Exercise 3.21. Propose the definition of $T_1 \wedge T_2 \wedge T_3 \wedge T_4$. Do this exercise before reading ahead.

We can also define $T_1 \wedge T_2 \wedge T_3$ in a more systematic (yet equivalent) way using symmetric groups. Let S_3 be the permutation group of $\{1,2,3\}$. An element $\sigma \in S_3$ is a bijective map $\sigma : \{1,2,3\} \to \{1,2,3\}$. For instance, a map satisfying $\sigma(1) = 2$, $\sigma(2) = 3$ and $\sigma(3) = 1$ is an example of an element in S_3. We can express this σ by:

$$\begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix} \quad \text{or simply:} \quad (123)$$

A map $\tau \in S_3$ given by $\tau(1) = 2$, $\tau(2) = 1$ and $\tau(3) = 3$ can be expressed as:

$$\begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix} \quad \text{or simply:} \quad (12)$$

This element, which switches two of the elements in $\{1,2,3\}$ and fixes the other one, is called a *transposition*.

Multiplication of two elements $\sigma_1, \sigma_2 \in S_3$ is defined by composition. Precisely, $\sigma_1\sigma_2$ is the composition $\sigma_1 \circ \sigma_2$. Note that this means the elements $\{1,2,3\}$ are input into σ_2 first, and then into σ_1. In general, $\sigma_1\sigma_2 \neq \sigma_2\sigma_1$. One can check easily that, for instance, we have:

$$\begin{aligned} (12)(23) &= (123) \\ (23)(12) &= (132) \end{aligned}$$

Elements in the permutation group S_n of n elements (usually denoted by $\{1, 2, \ldots, n\}$) can be represented and mutliplied in a similar way.

Exercise 3.22. Convince yourself that in S_5, we have:

$$(12345)(31) = (32)(145) = (32)(15)(14)$$

The above exercise shows that we can decompose $(12345)(31)$ into a product of three transpositions (32), (15) and (14). In fact, any element in S_n can be decomposed this way. Here we state a standard theorem in elementary group theory:

Theorem 3.22. *Every element $\sigma \in S_n$ can be expressed as a product of transpositions: $\sigma = \tau_1\tau_2 \ldots \tau_r$. Such a decomposition is not unique. However, if $\sigma = \widetilde{\tau}_1\widetilde{\tau}_2 \ldots \widetilde{\tau}_k$ is another decomposition of σ into transpositions, then we have $(-1)^k = (-1)^r$.*

Proof. Consult any standard textbook on Abstract Algebra. $\square$

In view of Theorem 3.22, given an element $\sigma \in S_n$ which can be decomposed into the product of r transpositions, we define:

$$\operatorname{sgn}(\sigma) := (-1)^r.$$

For instance, $\operatorname{sgn}(12345) = (-1)^3 = -1$, and $\operatorname{sgn}(123) = (-1)^2 = 1$. Certainly, if τ is a transposition, we have $\operatorname{sgn}(\sigma\tau) = -\operatorname{sgn}(\sigma)$.

Now we are ready to state an equivalent way to define triple wedge product using the above notations:

$$T_1 \wedge T_2 \wedge T_3 := \sum_{\sigma \in S_3} \operatorname{sgn}(\sigma) T_{\sigma(1)} \otimes T_{\sigma(2)} \otimes T_{\sigma(3)}.$$

We can verify that it gives the same expression as before:

$$\begin{aligned}
&\sum_{\sigma \in S_3} \operatorname{sgn}(\sigma) T_{\sigma(1)} \otimes T_{\sigma(2)} \otimes T_{\sigma(3)} \\
&= T_1 \otimes T_2 \otimes T_3 && \sigma = \mathrm{id} \\
&\quad - T_2 \otimes T_1 \otimes T_3 && \sigma = (12) \\
&\quad - T_3 \otimes T_2 \otimes T_1 && \sigma = (13) \\
&\quad - T_1 \otimes T_3 \otimes T_2 && \sigma = (23) \\
&\quad + T_2 \otimes T_3 \otimes T_1 && \sigma = (123) = (13)(12) \\
&\quad + T_3 \otimes T_1 \otimes T_2 && \sigma = (132) = (12)(13)
\end{aligned}$$

In general, we define:

Definition 3.23 (Wedge Product)**.** Let V be a finite dimensional vector space, and V^* be the dual space of V. Then, given any $T_1, \ldots, T_k \in V^*$, we define their *k-th wedge product* by:

$$T_1 \wedge \cdots \wedge T_k := \sum_{\sigma \in S_k} \operatorname{sgn}(\sigma)\, T_{\sigma(1)} \otimes \ldots \otimes T_{\sigma(k)}$$

where S_k is the permutation group of $\{1, \ldots, k\}$. The vector space spanned by $T_1 \wedge \cdots \wedge T_k$'s (where $T_1, \ldots, T_k \in V^*$) is denoted by $\wedge^k V^*$.

Remark 3.24. It is a convention to define $\wedge^0 V^* := \mathbb{R}$. □

If we switch any pair of the T_i's, then the wedge product differs by a minus sign. To show this, let $\tau \in S_k$ be a transposition, then for any $\sigma \in S_k$, we have $\operatorname{sgn}(\sigma \circ \tau) = -\operatorname{sgn}(\sigma)$. Therefore, we get:

$$\begin{aligned}
T_{\tau(1)} \wedge \cdots \wedge T_{\tau(k)} &= \sum_{\sigma \in S_k} \operatorname{sgn}(\sigma) T_{\sigma(\tau(1))} \otimes \ldots \otimes T_{\sigma(\tau(k))} \\
&= -\sum_{\sigma \in S_k} \operatorname{sgn}(\sigma \circ \tau) T_{\sigma\circ\tau(1)} \otimes \ldots \otimes T_{\sigma\circ\tau(k)} \\
&= -\sum_{\sigma \in S_k} \operatorname{sgn}(\sigma') T_{\sigma'(1)} \otimes \ldots \otimes T_{\sigma'\tau(k)} && (\text{where } \sigma' := \sigma \circ \tau) \\
&= -T_1 \wedge \cdots \wedge T_k.
\end{aligned}$$

The last step follows from the fact that $\sigma \mapsto \sigma \circ \tau$ is a bijection between S_k and itself.

Exercise 3.23. Write down $T_1 \wedge T_2 \wedge T_3 \wedge T_4$ explicitly in terms of tensor products (with no wedge and summation sign).

Exercise 3.24. Show that $\dim \wedge^k V^* = C^n_k$, when $n = \dim V$ and $0 \le k \le n$, by writing a basis for $\wedge^k V^*$. Show also that $\wedge^k V^* = \{0\}$ if $k > \dim V$.

Exercise 3.25. Let $\{\mathsf{e}_i\}_{i=1}^n$ be a basis for a vector space V, and $\{\mathsf{e}_i^*\}_{i=1}^n$ be the corresponding dual basis for V^*. Show that:

$$\left(\mathsf{e}_{i_1}^* \wedge \cdots \wedge \mathsf{e}_{i_k}^*\right)(\mathsf{e}_{j_1}, \ldots, \mathsf{e}_{j_k}) = \delta_{i_1 j_1} \cdots \delta_{i_k j_k}.$$

Remark 3.25. The vector space $\wedge^k V^*$ is *spanned* by $T_1 \wedge \cdots \wedge T_k$'s where $T_1, \ldots, T_k \in V^*$. Note that not all elements in V^* can be expressed in the form of $T_1 \wedge \cdots \wedge T_k$. For instance when $V = \mathbb{R}^4$ with standard basis $\{\mathsf{e}_i\}_{i=1}^4$, the element $\sigma = \mathsf{e}_1^* \wedge \mathsf{e}_2^* + \mathsf{e}_3^* \wedge \mathsf{e}_4^* \in \wedge^2 V^*$ cannot be written in the form of $T_1 \wedge T_2$ where $T_1, T_2 \in V^*$. It is because $(T_1 \wedge T_2) \wedge (T_1 \wedge T_2) = 0$ for any $T_1, T_2 \in V^*$, while $\sigma \wedge \sigma = 2\mathsf{e}_1^* \wedge \mathsf{e}_2^* \wedge \mathsf{e}_3^* \wedge \mathsf{e}_4^* \neq 0$. □

In the above remark, we take the wedge product between elements in $\wedge^2 V^*$. It is defined in a natural way that for any $T_1, \ldots T_k, S_1, \ldots, S_r \in V^*$, we have:

$$\underbrace{(T_1 \wedge \cdots \wedge T_k)}_{\in \wedge^k V^*} \wedge \underbrace{(S_1 \wedge \cdots \wedge S_r)}_{\in \wedge^r V^*} = \underbrace{T_1 \wedge \cdots \wedge T_k \wedge S_1 \wedge \cdots \wedge S_r}_{\in \wedge^{k+r} V^*}$$

and extended linearly to other elements in $\wedge^k V^*$ and $\wedge^r V^*$. For instance, we have:

$$\underbrace{(T_1 \wedge T_2 + S_1 \wedge S_2)}_{\in \wedge^2 V^*} \wedge \underbrace{\sigma}_{\in \wedge^k V^*} = \underbrace{T_1 \wedge T_2 \wedge \sigma + S_1 \wedge S_2 \wedge \sigma}_{\in \wedge^{k+2} V^*}.$$

While it is true that $T_1 \wedge T_2 = -T_2 \wedge T_1$ for any $T_1, T_2 \in V^*$, it is generally *not* true that $\sigma \wedge \eta = -\eta \wedge \sigma$ where $\sigma \in \wedge^k V^*$ and $\eta \in \wedge^r V^*$. For instance, let $T_1, \ldots, T_5 \in V^*$ and consider $\sigma = T_1 \wedge T_2$ and $\eta = T_3 \wedge T_4 \wedge T_5$. Then we can see that:

$$\begin{aligned}
\sigma \wedge \eta &= T_1 \wedge T_2 \wedge T_3 \wedge T_4 \wedge T_5 \\
&= -T_1 \wedge T_3 \wedge T_4 \wedge T_5 \wedge T_2 && \text{(switching } T_2 \text{ subsequently with } T_3, T_4, T_5) \\
&= T_3 \wedge T_4 \wedge T_5 \wedge T_1 \wedge T_2 && \text{(switching } T_1 \text{ subsequently with } T_3, T_4, T_5) \\
&= \eta \wedge \sigma.
\end{aligned}$$

Proposition 3.26. *Let V be a finite dimensional vector space, and V^* be the dual space of V. Given any $\sigma \in \wedge^k V^*$ and $\eta \in \wedge^r V^*$, we have:*

$$\sigma \wedge \eta = (-1)^{kr} \eta \wedge \sigma. \tag{3.5}$$

Clearly from (3.5), *any $\omega \in \wedge^{\text{even}} V^*$ commutes with any $\sigma \in \wedge^k V^*$.*

Proof. By linearity, it suffices to prove that case $\sigma = T_1 \wedge \cdots \wedge T_k$ and $\eta = S_1 \wedge \cdots \wedge S_r$ where $T_i, S_j \in V^*$, in which we have:

$$\sigma \wedge \eta = T_1 \wedge \cdots \wedge T_k \wedge S_1 \wedge \cdots \wedge S_r$$

In order to switch all the T_i's with the S_j's, we can first switch T_k subsequently with each of $S_1, \ldots, S_r$ and each switching contributes to a factor of (-1). Precisely, we have:

$$T_1 \wedge \cdots \wedge T_k \wedge S_1 \wedge \cdots \wedge S_r = (-1)^r T_1 \wedge \cdots \wedge T_{k-1} \wedge S_1 \wedge \cdots \wedge S_r \wedge T_k.$$

By repeating this sequence of switching on each of T_{k-1}, T_{k-2}, etc., we get a factor of $(-1)^r$ for each set of switching, and so we finally get the following as desired:

$$T_1 \wedge \cdots \wedge T_k \wedge S_1 \wedge \cdots \wedge S_r = [(-1)^r]^k \, S_1 \wedge \cdots \wedge S_r \wedge T_1 \wedge \cdots \wedge T_k$$

□

From Exercise 3.24, we know that $\dim \wedge^n V^* = 1$ if $n = \dim V$. In fact, every element $\sigma \in \dim \wedge^n V^*$ is a constant multiple of $\mathsf{e}_1^* \wedge \cdots \wedge \mathsf{e}_n^*$, and it is interesting (and important) to note that this constant multiple is related to a determinant! Precisely, for each $i = 1, \ldots, n$, we consider the elements:

$$\omega_i = \sum_{j=1}^{n} a_{ij} \mathsf{e}_j^* \in V^*$$

where a_{ij} are real constants. Then, the wedge product of all ω_i's are given by:

$$\begin{aligned}
\omega_1 \wedge \cdots \wedge \omega_n &= \left(\sum_{j_1=1}^{n} a_{1j_1} \mathsf{e}_{j_1}^*\right) \wedge \left(\sum_{j_2=1}^{n} a_{2j_2} \mathsf{e}_{j_2}^*\right) \wedge \cdots \wedge \left(\sum_{j_n=1}^{n} a_{nj_n} \mathsf{e}_{j_n}^*\right) \\
&= \sum_{j_1, \ldots, j_n \text{ distinct}} a_{1j_1} a_{2j_2} \ldots a_{nj_n} \mathsf{e}_{j_1}^* \wedge \cdots \wedge \mathsf{e}_{j_n}^* \\
&= \sum_{\sigma \in S_n} a_{1\sigma(1)} a_{2\sigma(2)} \ldots a_{n\sigma(n)} \mathsf{e}_{\sigma(1)}^* \wedge \cdots \wedge \mathsf{e}_{\sigma(n)}^*
\end{aligned}$$

Next we want to find a relation between $\mathsf{e}_{\sigma(1)}^* \wedge \cdots \wedge \mathsf{e}_{\sigma(n)}^*$ and $\mathsf{e}_1^* \wedge \cdots \wedge \mathsf{e}_n^*$. $\sigma \in S_n$, by decomposing it into transpositions $\sigma = \tau_1 \circ \cdots \circ \tau_k$, then we have:

$$\begin{aligned}
\mathsf{e}_{\sigma(1)}^* \wedge \cdots \wedge \mathsf{e}_{\sigma(n)}^* &= \mathsf{e}_{\tau_1 \circ \cdots \circ \tau_k(1)}^* \wedge \cdots \wedge \mathsf{e}_{\tau_1 \circ \cdots \circ \tau_k(n)}^* \\
&= (-1) \mathsf{e}_{\tau_2 \circ \cdots \circ \tau_k(1)}^* \wedge \cdots \wedge \mathsf{e}_{\tau_2 \circ \cdots \circ \tau_k(n)}^* \\
&= (-1)^2 \mathsf{e}_{\tau_3 \circ \cdots \circ \tau_k(1)}^* \wedge \cdots \wedge \mathsf{e}_{\tau_3 \circ \cdots \circ \tau_k(n)}^* \\
&= \ldots \\
&= (-1)^{k-1} \mathsf{e}_{\tau_k(1)}^* \wedge \cdots \wedge \mathsf{e}_{\tau_k(n)}^* \\
&= (-1)^k \mathsf{e}_1^* \wedge \cdots \wedge \mathsf{e}_n^* \\
&= \operatorname{sgn}(\sigma) \mathsf{e}_1^* \wedge \cdots \wedge \mathsf{e}_n^*.
\end{aligned}$$

Therefore, we have:

$$\omega_1 \wedge \cdots \wedge \omega_n = \left(\sum_{\sigma \in S_n} \operatorname{sgn}(\sigma) a_{1\sigma(1)} a_{2\sigma(2)} \ldots a_{n\sigma(n)}\right) \mathsf{e}_1^* \wedge \cdots \wedge \mathsf{e}_n^*.$$

Note that the sum:

$$\sum_{\sigma \in S_n} \operatorname{sgn}(\sigma) a_{1\sigma(1)} a_{2\sigma(2)} \ldots a_{n\sigma(n)}$$

is exactly the determinant of the matrix A whose (i, j)-th entry is a_{ij}. To summarize, let's state it as a proposition:

Proposition 3.27. *Let V^* be the dual space of a vector space V of dimension n, and let $\{\mathsf{e}_i\}_{i=1}^n$ be a basis for V, and $\{\mathsf{e}_i^*\}_{i=1}^n$ be the corresponding dual basis for V^*. Given any n elements $\omega_i = \sum_{j=1}^{n} a_{ij} \mathsf{e}_j^* \in V^*$, we have:*

$$\omega_1 \wedge \cdots \wedge \omega_n = (\det A)\ \mathsf{e}_1^* \wedge \cdots \wedge \mathsf{e}_n^*,$$

where A is the $n \times n$ matrix whose (i, j)-th entry is a_{ij}.

Exercise 3.26. Given an n-dimensional vector space V. Show that $\omega_1, \ldots, \omega_n \in V^*$ are linearly independent if and only if $\omega_1 \wedge \cdots \wedge \omega_n \neq 0$.

Exercise 3.27. Generalize Proposition 3.27. Precisely, now given

$$\omega_i = \sum_{j=1}^{n} a_{ij} \mathsf{e}_j^* \in V^*$$

where $1 \le i \le k < \dim V$, express $\omega_1 \wedge \cdots \wedge \omega_k$ in terms of e_i^*'s.

Exercise 3.28. Regard $\det : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ as a multilinear map:

$$\det(\mathsf{v}_1, \ldots, \mathsf{v}_n) := \begin{vmatrix} | & & | \\ \mathsf{v}_1 & \cdots & \mathsf{v}_n \\ | & & | \end{vmatrix}.$$

Denote $\{\mathsf{e}_i\}$ the standard basis for $\mathbb{R}^n$. Show that:

$$\det = \mathsf{e}_1^* \wedge \cdots \wedge \mathsf{e}_n^*.$$

3.4.2. Differential Forms on Smooth Manifolds. In the simplest term, differential forms on a smooth manifold are wedge products of cotangent vectors in T^*M. At each point $p \in M$, let $(u_1, \ldots, u_n)$ be the local coordinates near p, then the cotangent space T_p^*M is spanned by $\left\{ du^1\big|_p, \ldots, du^n\big|_p \right\}$, and a smooth differential 1-form α is a map from M to T^*M such that it can be locally expressed as:

$$\alpha(p) = \left(p, \sum_{i=1}^{n} \alpha_i(p)\, du^i\Big|_p \right)$$

where α_i are smooth functions locally defined near p. Since the based point p can usually be understood from the context, we usually denote α by simply:

$$\alpha = \sum_{i=1}^{n} \alpha_i\, du^i.$$

Since T_p^*M is a finite dimensional vector space, we can consider the wedge products of its elements. A *differential k-form* ω on a smooth manifold M is a map which assigns each point $p \in M$ to an element in $\wedge^k T_p^*M$. Precisely:

Definition 3.28 (Smooth Differential k-Forms)**.** Let M be a smooth manifold. A *smooth differential k-form* ω on M is a map $\omega_p : \underbrace{T_pM \times \ldots \times T_pM}_{k \text{ times}} \to \mathbb{R}$ at each $p \in M$ such that under any local parametrization $\mathsf{F}(u_1, \ldots, u_n) : \mathcal{U} \to M$, it can be written in the form:

$$\omega = \sum_{i_1, \ldots, i_k = 1}^{n} \omega_{i_1 i_2 \cdots i_k}\, du^{i_1} \wedge \cdots \wedge du^{i_k}$$

where $\omega_{i_1 i_2 \ldots i_k}$'s are smooth scalar functions locally defined in $\mathsf{F}(\mathcal{U})$, and they are commonly called the *local components* of ω. The vector space of all smooth differential k-forms on M is denoted by $\wedge^k T^*M$.

Remark 3.29. It is a convention to denote $\wedge^0 T^*M := C^\infty(M, \mathbb{R})$, the vector space of all smooth scalar functions defined on M. $\square$

We will mostly deal with differential k-forms that are smooth. Therefore, we will very often call a *smooth differential k-form* simply by a *differential k-form*, or even simpler, a *k-form*. As we will see in the next section, the language of differential forms will unify and generalize the curl, grad and div in Multivariable Calculus and Physics courses.

From algebraic viewpoint, the manipulations of differential k-forms on a manifold are similar to those for wedge products of a finite-dimensional vector space. The major difference is a manifold is usually covered by more than one local parametrizations, hence there are conversion rules for differential k-forms from one local coordinate system to another.

Example 3.30. Consider $\mathbb{R}^2$ with (x, y) and (r, θ) as its two local coordinates. Given a 2-form $\omega = dx \wedge dy$, for instance, we can express it in terms of the polar coordinates (r, θ):

$$\begin{aligned} dx &= \frac{\partial x}{\partial r}\,dr + \frac{\partial x}{\partial \theta}\,d\theta \\ &= (\cos\theta)\,dr - (r\sin\theta)\,d\theta \\ dy &= \frac{\partial y}{\partial r}\,dr + \frac{\partial y}{\partial \theta}\,d\theta \\ &= (\sin\theta)\,dr + (r\cos\theta)\,d\theta \end{aligned}$$

Therefore, using $dr \wedge dr = 0$ and $d\theta \wedge d\theta = 0$, we get:

$$\begin{aligned} dx \wedge dy &= (r\cos^2\theta)dr \wedge d\theta - (r\sin^2)d\theta \wedge dr \\ &= (r\cos^2\theta + r\sin^2\theta)\,dr \wedge d\theta \\ &= r\,dr \wedge d\theta. \end{aligned}$$

□

Exercise 3.29. Define a 2-form on $\mathbb{R}^3$ by:

$$\omega = x\,dy \wedge dz + y\,dz \wedge dx + z\,dx \wedge dy.$$

Express ω in terms of spherical coordinates (ρ, θ, φ), defined by:

$$(x, y, z) = (\rho\sin\varphi\cos\theta, \rho\sin\varphi\sin\theta, \rho\cos\varphi).$$

Exercise 3.30. Let ω be the 2-form on $\mathbb{R}^{2n}$ given by:

$$\omega = dx^1 \wedge dx^2 + dx^3 \wedge dx^4 + \ldots + dx^{2n-1} \wedge dx^{2n}.$$

Compute $\underbrace{\omega \wedge \cdots \wedge \omega}_{n \text{ times}}$.

Exercise 3.31. Let $(u_1, \ldots, u_n)$ and $(v_1, \ldots, v_n)$ be two local coordinates of a smooth manifold M. Show that:

$$du^1 \wedge \cdots \wedge du^n = \det \frac{\partial(u_1, \ldots, u_n)}{\partial(v_1, \ldots, v_n)}\,dv^1 \wedge \cdots \wedge dv^n.$$

Exercise 3.32. Show that on $\mathbb{R}^3$, a $(2,0)$-tensor T is in $\wedge^2(\mathbb{R}^2)^*$ if and only if $T(\mathsf{v}, \mathsf{v}) = 0$ for any $\mathsf{v} \in \mathbb{R}^3$.

3.5. Exterior Derivatives

Exterior differentiation is an important operations on differential forms. It not only generalizes and unifies the curl, grad, div operators in Multivariable Calculus and Physics, but also leads to the development of de Rham cohomology to be discussed in Chapter 5.

3.5.1. Definition of Exterior Derivatives. Exterior differentiation, commonly denoted by the symbol d, takes a k-form to a $(k+1)$-form. To begin, let's define it on scalar functions first. Suppose $(u_1, \ldots, u_n)$ are local coordinates of M^n, then given any smooth scalar function $f \in C^\infty(M, \mathbb{R})$, we define:

$$df := \sum_{i=1}^{n} \frac{\partial f}{\partial u_i}\, du^i \tag{3.6}$$

Although (3.6) involves local coordinates, it can be easily shown that df is independent of local coordinates. Suppose $(v_1, \ldots, v_n)$ is another local coordinates of M which overlap with $(u_1, \ldots, u_n)$. By the chain rule, we have:

$$\begin{aligned} \frac{\partial f}{\partial u_i} &= \sum_{k=1}^{n} \frac{\partial f}{\partial v_k} \frac{\partial v_k}{\partial u_i} \\ dv^k &= \sum_{i=1}^{n} \frac{\partial v_k}{\partial u_i}\, du^i \end{aligned}$$

which combine to give:

$$\sum_{i=1}^{n} \frac{\partial f}{\partial u_i}\, du^i = \sum_{i=1}^{n} \sum_{k=1}^{n} \frac{\partial f}{\partial v_k} \frac{\partial v_k}{\partial u_i}\, du^i = \sum_{k=1}^{n} \frac{\partial f}{\partial v_k}\, dv^k.$$

Therefore, if f is smooth on M then df is a smooth 1-form on M. The components of df are $\frac{\partial f}{\partial u_i}$'s, and so df is analogous to ∇f in Multivariable Calculus. Note that as long as f is C^∞ just in an open set $\mathcal{U} \subset M$, we can also define df locally on $\mathcal{U}$ since (3.6) is a local expression.

Exterior derivatives can also be defined on differential forms of higher degrees. Let $\alpha \in \wedge^1 T^* M$, which can be locally written as:

$$\alpha = \sum_{i=1}^{n} \alpha_i\, du^i$$

where α_i's are smooth functions locally defined in a local coordinate chart. Then, we define:

$$d\alpha := \sum_{i=1}^{n} d\alpha_i \wedge du^i = \sum_{i=1}^{n} \sum_{j=1}^{n} \frac{\partial \alpha_i}{\partial u_j}\, du^j \wedge du^i. \tag{3.7}$$

Using the fact that $du^j \wedge du^i = -du^i \wedge du^j$ and $du^i \wedge du^i = 0$, we can also express $d\alpha$ as:

$$d\alpha = \sum_{1 \le j < i \le n} \left(\frac{\partial \alpha_i}{\partial u_j} - \frac{\partial \alpha_j}{\partial u_i} \right) du^j \wedge du^i.$$

Example 3.31. Take $M = \mathbb{R}^3$ as an example, and let (x, y, z) be the (usual) coordinates of $\mathbb{R}^3$, then given any 1-form $\alpha = P\,dx + Q\,dy + R\,dz$ (which is analogous to the vector field $P\mathsf{i} + Q\mathsf{j} + R\mathsf{k}$), we have:

$$\begin{aligned}
d\alpha &= dP \wedge dx + dQ \wedge dy + dR \wedge dz \\
&= \left(\frac{\partial P}{\partial x}\,dx + \frac{\partial P}{\partial y}\,dy + \frac{\partial P}{\partial z}\,dz\right) \wedge dx + \left(\frac{\partial Q}{\partial x}\,dx + \frac{\partial Q}{\partial y}\,dy + \frac{\partial Q}{\partial z}\,dz\right) \wedge dy \\
&\quad + \left(\frac{\partial R}{\partial x}\,dx + \frac{\partial R}{\partial y}\,dy + \frac{\partial R}{\partial z}\,dz\right) \wedge dz \\
&= \frac{\partial P}{\partial y}\,dy \wedge dx + \frac{\partial P}{\partial z}\,dz \wedge dx + \frac{\partial Q}{\partial x}\,dx \wedge dy + \frac{\partial Q}{\partial z}\,dz \wedge dy \\
&\quad + \frac{\partial R}{\partial x}\,dx \wedge dz + \frac{\partial R}{\partial y}\,dy \wedge dz \\
&= \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx \wedge dy - \left(\frac{\partial R}{\partial x} - \frac{\partial P}{\partial z}\right) dz \wedge dx + \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right) dy \wedge dz
\end{aligned}$$

which is analogous to $\nabla \times (P\mathsf{i} + Q\mathsf{j} + R\mathsf{k})$ by declaring the correspondence $\{\mathsf{i}, \mathsf{j}, \mathsf{k}\}$ with $\{dy \wedge dz, dz \wedge dx, dx \wedge dy\}$. □

One can check that the definition of $d\alpha$ stated in (3.7) is independent of local coordinates. On general k-forms, the exterior derivatives are defined in a similar way as:

Definition 3.32 (Exterior Derivatives)**.** Let M^n be a smooth manifold and $(u_1, \ldots, u_n)$ be local coordinates on M. Given any (smooth) k-form

$$\omega = \sum_{j_1,\ldots,j_k=1}^{n} \omega_{j_1\cdots j_k}\, du^{j_1} \wedge \cdots \wedge du^{j_k},$$

we define:

$$d\omega := \sum_{j_1,\cdots,j_k=1}^{n} d\omega_{j_1\cdots j_k} \wedge du^{j_1} \wedge \cdots \wedge du^{j_k} \tag{3.8}$$

$$= \sum_{j_1,\cdots,j_k=1}^{n} \sum_{i=1}^{n} \frac{\partial \omega_{j_1\cdots j_k}}{\partial u_i} du^i \wedge du^{j_1} \wedge \cdots \wedge du^{j_k}$$

In particular, if ω is an n-form (where $n = \dim V$), we have $d\omega = 0$.

Exercise 3.33. Show that $d\omega$ defined as in (3.8) does not depend on the choice of local coordinates.

Example 3.33. Consider $\mathbb{R}^2$ equipped with polar coordinates (r, θ). Consider the 1-form:

$$\omega = (r \sin\theta)\, dr.$$

Then, we have

$$\begin{aligned}
d\omega &= \frac{\partial (r\sin\theta)}{\partial r}\, dr \wedge dr + \frac{\partial (r\sin\theta)}{\partial \theta}\, d\theta \wedge dr \\
&= 0 + (r\cos\theta)\, d\theta \wedge dr \\
&= -(r\cos\theta)\, dr \wedge d\theta.
\end{aligned}$$

□

Exercise 3.34. Let $\omega = F_1\, dy \wedge dz + F_2\, dz \wedge dx + F_3\, dx \wedge dy$ be a smooth 2-form on $\mathbb{R}^3$. Compute $d\omega$. What operator in Multivariable Calculus is the d analogous to in this case?

Exercise 3.35. Let ω, η, θ be the following differential forms on $\mathbb{R}^3$:

$$\begin{aligned}\omega &= x\, dx - y, dy\\ \eta &= z\, dx \wedge dy + x\, dy \wedge dz\\ \theta &= z\, dy\end{aligned}$$

Compute: $\omega \wedge \eta$, $\omega \wedge \eta \wedge \theta$, $d\omega$, $d\eta$ and $d\theta$.

3.5.2. Properties of Exterior Derivatives. The exterior differentiation d can hence be regarded as a chain of maps:

$$\wedge^0 T^*M \xrightarrow{d} \wedge^1 T^*M \xrightarrow{d} \wedge^2 T^*M \xrightarrow{d} \cdots \xrightarrow{d} \wedge^{n-1} T^*M \xrightarrow{d} \wedge^n T^*M.$$

Here we abuse the use of the symbol d a little bit – we use the same symbol d for all the maps $\wedge^k T^*M \xrightarrow{d} \wedge^{k+1} T^*M$ in the chain. The following properties about exterior differentiation are not difficult to prove:

Proposition 3.34. *For any k-forms ω and η, and any smooth scalar function f, we have the following:*

(1) $d(\omega + \eta) = d\omega + d\eta$

(2) $d(f\omega) = df \wedge \omega + f\, d\omega$

Proof. (1) is easy to prove (left as an exercise for readers). To prove (2), we consider local coordinates $(u_1, \ldots, u_n)$ and let $\omega = \sum_{j_1,\ldots,j_k=1}^{n} \omega_{j_1\cdots j_k}\, du^{j_1} \wedge \cdots \wedge du^{j_k}$. Then, we have:

$$\begin{aligned} d(f\omega) &= \sum_{j_1,\ldots,j_k=1}^{n} \sum_{i=1}^{n} \frac{\partial}{\partial u_i}(f\omega_{j_1\cdots j_k})\, du^i \wedge du^{j_1} \wedge \cdots \wedge du^{j_k}\\ &= \sum_{j_1,\ldots,j_k=1}^{n} \sum_{i=1}^{n} \left(\frac{\partial f}{\partial u_i}\, \omega_{j_1\cdots j_k} + f\, \frac{\partial \omega_{j_1\cdots j_k}}{\partial u_i}\right) du^i \wedge du^{j_1} \wedge \cdots \wedge du^{j_k}\\ &= \left(\sum_{i=1}^{n} \frac{\partial f}{\partial u_i}\, du^i\right) \wedge \left(\sum_{j_1,\ldots,j_k=1}^{n} \omega_{j_1\cdots j_k}\, du^{j_1} \wedge \cdots \wedge du^{j_k}\right)\\ &\quad + f \sum_{j_1,\cdots,j_k=1}^{n} \sum_{i=1}^{n} \frac{\partial \omega_{j_1\cdots j_k}}{\partial u_i} du^i \wedge du^{j_1} \wedge \cdots \wedge du^{j_k} \end{aligned}$$

as desired. □

Identity (2) in Proposition 3.34 can be regarded as a kind of product rule. Given a k-form α and a r-form β, the general product rule for exterior derivative is stated as:

Proposition 3.35. *Let $\alpha \in \wedge^k T^*M$ and $\beta \in \wedge^r T^*M$ be smooth differential forms on M, then we have:*

$$d(\alpha \wedge \beta) = d\alpha \wedge \beta + (-1)^k\, \alpha \wedge d\beta.$$

Exercise 3.36. Prove Proposition 3.35. Based on your proof, explain briefly why the product rule does not involve any factor of $(-1)^r$.

Exercise 3.37. Given three differential forms α, β and γ such that $d\alpha = 0$, $d\beta = 0$ and $d\gamma = 0$. Show that:
$$d(\alpha \wedge \beta \wedge \gamma) = 0.$$

An crucial property of exterior derivatives is that the *composition* is zero. For instance, given a smooth scalar function $f(x,y,z)$ defined on $\mathbb{R}^3$, we have:
$$df = \frac{\partial f}{\partial x}\,dx + \frac{\partial f}{\partial y}\,dy + \frac{\partial f}{\partial z}\,dz.$$
Taking exterior derivative one more time, we get:
$$\begin{aligned}
d(df) &= \left(\frac{\partial}{\partial x}\frac{\partial f}{\partial x}\,dx + \frac{\partial}{\partial y}\frac{\partial f}{\partial x}\,dy + \frac{\partial}{\partial z}\frac{\partial f}{\partial x}\,dz\right) \wedge dx \\
&\quad + \left(\frac{\partial}{\partial x}\frac{\partial f}{\partial y}\,dx + \frac{\partial}{\partial y}\frac{\partial f}{\partial y}\,dy + \frac{\partial}{\partial z}\frac{\partial f}{\partial y}\,dz\right) \wedge dy \\
&\quad + \left(\frac{\partial}{\partial x}\frac{\partial f}{\partial z}\,dx + \frac{\partial}{\partial y}\frac{\partial f}{\partial z}\,dy + \frac{\partial}{\partial z}\frac{\partial f}{\partial z}\,dz\right) \wedge dz \\
&= \left(\frac{\partial}{\partial x}\frac{\partial f}{\partial y} - \frac{\partial}{\partial y}\frac{\partial f}{\partial x}\right)\,dx \wedge dy + \left(\frac{\partial}{\partial z}\frac{\partial f}{\partial x} - \frac{\partial}{\partial x}\frac{\partial f}{\partial z}\right)\,dz \wedge dx \\
&\quad + \left(\frac{\partial}{\partial y}\frac{\partial f}{\partial z} - \frac{\partial}{\partial z}\frac{\partial f}{\partial y}\right)\,dy \wedge dz
\end{aligned}$$
Since partial derivatives commute, we get $d(df) = 0$, or in short $d^2 f = 0$, for any scalar function f. The fact that $d^2 = 0$ is generally true on smooth differential forms, not only for scalar functions. Precisely, we have:

Proposition 3.36. *Let ω be a smooth k-form defined on a smooth manifold M, then we have:*
$$d^2\omega := d(d\omega) = 0.$$

Proof. Let $\omega = \sum_{j_1,\ldots,j_k=1}^{n} \omega_{j_1\cdots j_k}\,du^{j_1} \wedge \cdots \wedge du^{j_k}$, then:
$$d\omega = \sum_{j_1,\cdots,j_k=1}^{n} \sum_{i=1}^{n} \frac{\partial \omega_{j_1\cdots j_k}}{\partial u_i}\,du^i \wedge du^{j_1} \wedge \cdots \wedge du^{j_k}.$$
$$\begin{aligned}
d^2\omega &= d\left(\sum_{j_1,\cdots,j_k=1}^{n} \sum_{i=1}^{n} \frac{\partial \omega_{j_1\cdots j_k}}{\partial u_i}\,du^i \wedge du^{j_1} \wedge \cdots \wedge du^{j_k}\right) \\
&= \sum_{j_1,\cdots,j_k=1}^{n} \sum_{i=1}^{n} \sum_{l=1}^{n} \frac{\partial^2 \omega_{j_1\ldots j_k}}{\partial u_l \partial u_i}\,du^l \wedge du^i \wedge du^{j_1} \wedge \cdots \wedge du^{j_k}
\end{aligned}$$
For each fixed k-tuple $(j_1,\ldots,j_k)$, the term $\sum_{i,l=1}^{n} \frac{\partial^2 \omega_{j_1\ldots j_k}}{\partial u_l \partial u_i}\,du^l \wedge du^i$ can be rewritten as:
$$\sum_{1 \le i < l \le n} \left(\frac{\partial^2 \omega_{j_1\ldots j_k}}{\partial u_l \partial u_i} - \frac{\partial^2 \omega_{j_1\ldots j_k}}{\partial u_i \partial u_l}\right)\,du^l \wedge du^i$$
which is zero since partial derivatives commute. It concludes that $d^2\omega = 0$. $\square$

Proposition 3.36 is a important fact that leads to the development of de Rham cohomology in Chapter 5.

In Multivariable Calculus, we learned that given a vector field $\mathsf{F} = P\mathsf{i} + Q\mathsf{j} + R\mathsf{k}$ and a scalar function f, we have:

$$\nabla \times \nabla f = 0$$
$$\nabla \cdot (\nabla \times \mathsf{F}) = 0$$

These two formulae can be unified using the language of differential forms. The one-form df corresponds to the vector field ∇f:

$$df = \frac{\partial f}{\partial x}\,dx + \frac{\partial f}{\partial y}\,dy + \frac{\partial f}{\partial z}\,dz$$
$$\nabla f = \frac{\partial f}{\partial x}\mathsf{i} + \frac{\partial f}{\partial y}\mathsf{j} + \frac{\partial f}{\partial z}\mathsf{k}$$

Define a one-form $\omega = P\,dx + Q\,dy + R\,dz$ on $\mathbb{R}^3$, which corresponds to the vector field F, then we have discussed that $d\omega$ corresponds to taking curl of F:

$$d\omega = \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx \wedge dy - \left(\frac{\partial R}{\partial x} - \frac{\partial P}{\partial z}\right) dz \wedge dx + \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right) dy \wedge dz$$
$$\nabla \times \mathsf{F} = \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\mathsf{k} - \left(\frac{\partial R}{\partial x} - \frac{\partial P}{\partial z}\right)\mathsf{j} + \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right)\mathsf{i}$$

If one takes $\omega = df$, and $\mathsf{F} = \nabla f$, then we have $d\omega = d(df) = 0$, which corresponds to the fact that $\nabla \times \mathsf{G} = \nabla \times \nabla f = 0$ in Multivariable Calculus.

Taking exterior derivative on a two-form $\beta = A\,dy \wedge dz + B\,dz \wedge dx + C\,dx \wedge dy$ corresponds to taking the divergence on the vector field $\mathsf{G} = A\mathsf{i} + B\mathsf{j} + C\mathsf{k}$ according to Exercise 3.34:

$$d\beta = \left(\frac{\partial A}{\partial x} + \frac{\partial B}{\partial y} + \frac{\partial C}{\partial z}\right) dx \wedge dy \wedge dz$$
$$\nabla \cdot \mathsf{G} = \left(\frac{\partial A}{\partial x} + \frac{\partial B}{\partial y} + \frac{\partial C}{\partial z}\right)$$

By taking $\beta = d\omega$, and $\mathsf{G} = \nabla \times \mathsf{F}$, then we have $d\beta = d(d\omega) = 0$ corresponding to $\nabla \cdot \mathsf{G} = \nabla \cdot (\nabla \times \mathsf{F}) = 0$ in Multivariable Calculus.

Here is a summary of the correspondences:

Differential Form on $\mathbb{R}^3$	Multivariable Calculus
$f(x,y,z)$	$f(x,y,z)$
$\omega = P\,dx + Q\,dy + R\,dz$	$\mathsf{F} = P\mathsf{i} + Q\mathsf{j} + R\mathsf{k}$
$\beta = A\,dy \wedge dz + B\,dz \wedge dx + C\,dx \wedge dy$	$\mathsf{G} = A\mathsf{i} + B\mathsf{j} + C\mathsf{k}$
df	∇f
$d\omega$	$\nabla \times \mathsf{F}$
$d\beta$	$\nabla \cdot \mathsf{G}$
$d^2 f = 0$	$\nabla \times \nabla f = 0$
$d^2\omega = 0$	$\nabla \cdot (\nabla \times \mathsf{F}) = 0$

3.5.3. Exact and Closed Forms. In Multivariable Calculus, we discussed various concepts of vector fields including potential functions, conservative vector fields, solenoidal vector fields, curl-less and divergence-less vector fields, etc. All these concepts can be unified using the language of differential forms.

As a reminder, a conservative vector field F is one that can be expressed as $\mathsf{F} = \nabla f$ where f is a scalar function. It is equivalent to saying that the 1-form ω can be expressed as $\omega = df$. Moreover, a solenoidal vector field G is one that can be expressed as $\mathsf{G} = \nabla \times \mathsf{F}$ for some vector field F. It is equivalent to saying that the 2-form β can be expressed as $\beta = d\omega$ for some 1-form ω.

Likewise, a curl-less vector field F (i.e. $\nabla \times \mathsf{F} = 0$) corresponds to a 1-form ω satisfying $d\omega = 0$; and a divergence-less vector field G (i.e. $\nabla \cdot \mathsf{G} = 0$) corresponds to a 2-form β satisfying $d\beta = 0$.

In view of the above correspondence, we introduce two terminologies for differential forms, namely *exact-ness* and *closed-ness*:

Definition 3.37 (Exact and Closed Forms)**.** Let ω be a smooth k-form defined on a smooth manifold M, then we say:

- ω is *exact* if there exists a $(k-1)$-form η defined on M such that $\omega = d\eta$;
- ω is *closed* if $d\omega = 0$.

Remark 3.38. By the fact that $d^2 = 0$ (Proposition 3.36), it is clear that every exact form is a closed form (but *not* vice versa). □

The list below showcases the corresponding concepts of exact/closed forms in Multivariable Calculus.

Differential Form on $\mathbb{R}^3$	Multivariable Calculus
exact 1-form	conservative vector field
closed 1-form	curl-less vector field
exact 2-form	solenoidal vector field
closed 2-form	divergence-less vector field

Example 3.39. On $\mathbb{R}^3$, the 1-form:

$$\alpha = yz\,dx + zx\,dy + xy\,dz$$

is exact since $\alpha = df$ where $f(x,y,z) = xyz$. By Proposition 3.36, we immediately get $d\alpha = d(df) = 0$, so α is a closed form. One can also verify this directly:

$$\begin{aligned} d\alpha &= (z\,dy + y\,dz) \wedge dx + (z\,dx + x\,dz) \wedge dy + (y\,dx + x\,dy) \wedge dz \\ &= (z - z)\,dx \wedge dy + (y - y)\,dz \wedge dx + (x - x)\,dy \wedge dz = 0. \end{aligned}$$

□

Example 3.40. The 1-form:

$$\alpha := -\frac{y}{x^2 + y^2}\,dx + \frac{x}{x^2 + y^2}\,dy$$

defined on $\mathbb{R}^2 \backslash \{(0,0)\}$ is closed:

$$\begin{aligned} d\alpha &= \frac{\partial}{\partial y}\left(-\frac{y}{x^2 + y^2}\right) dy \wedge dx + \frac{\partial}{\partial x}\left(\frac{x}{x^2 + y^2}\right) dx \wedge dy \\ &= \frac{y^2 - x^2}{(x^2 + y^2)^2}\,dy \wedge dx + \frac{y^2 - x^2}{(x^2 + y^2)^2}\,dx \wedge dy \\ &= 0 \end{aligned}$$

as $dx \wedge dy = -dy \wedge dx$. However, we will later see that α is not exact.

Note that even though we have $\alpha = df$ where $f(x,y) = \tan^{-1}\frac{y}{x}$, such an f is NOT smooth on $\mathbb{R}^2 \backslash \{(0,0)\}$. In order to claim α is exact, we require such an f to be smooth on the domain of α. □

Exercise 3.38. Consider the forms ω, η and θ on $\mathbb{R}^3$ defined in Exercise 3.35. Determine whether each of them is closed and/or exact on $\mathbb{R}^3$.

Exercise 3.39. The purpose of this exercise is to show that any closed 1-form ω on $\mathbb{R}^3$ must be exact. Let

$$\omega = P(x,y,z)\,dx + Q(x,y,z)\,dy + R(x,y,z)\,dz$$

be a closed 1-form on $\mathbb{R}^3$. Define $f : \mathbb{R}^3 \to \mathbb{R}$ by:

$$f(x,y,z) = \int_{t=0}^{t=1} \left(xP(tx,ty,tz) + yQ(tx,ty,tz) + zR(tx,ty,tz)\right)\,dt$$

Show that $\omega = df$. Point out exactly where you have used the fact that $d\omega = 0$.

3.5.4. Pull-Back of Tensors. Let's first begin by reviewing the *push-forward* and *pull-back* of tangent and cotangent vectors. Given a smooth map $\Phi : M \to N$ between two smooth manifolds M^m and N^n, its tangent map Φ_* takes a tangent vector in T_pM to a tangent vector in $T_{\Phi(p)}N$. If we let $\mathsf{F}(u_1,\ldots,u_m)$ be local coordinates of M, $\mathsf{G}(v_1,\ldots,v_n)$ be local coordinates of N and express the map Φ locally as:

$$(v_1,\ldots,v_n) = \mathsf{G}^{-1} \circ \Phi \circ \mathsf{F}(u_1,\ldots,u_m),$$

then Φ_* acts on the basis vectors $\left\{\dfrac{\partial}{\partial u_i}\right\}$ by:

$$\Phi_*\left(\frac{\partial}{\partial u_i}\right) = \frac{\partial \Phi}{\partial u_i} = \sum_j \frac{\partial v_j}{\partial u_i}\frac{\partial}{\partial v_j}.$$

The tangent map Φ_* is also commonly called the *push-forward* map. It is important to note that the v_j's in the partial derivatves $\frac{\partial v_j}{\partial u_i}$ can sometimes cause confusion if we talk about the push-forwards of two different smooth maps $\Phi : M \to N$ and $\Psi : M \to N$. Even with the same input $(u_1,\ldots,u_m)$, the output $\Phi(u_1,\ldots,u_m)$ and $\Psi(u_1,\ldots,u_m)$ are generally different and have different v_j-coordinates. To avoid this confusion, it is best to write:

$$\Phi_*\left(\frac{\partial}{\partial u_i}\right) = \sum_j \frac{\partial(v_j \circ \Phi)}{\partial u_i}\frac{\partial}{\partial v_j}$$

$$\Psi_*\left(\frac{\partial}{\partial u_i}\right) = \sum_j \frac{\partial(v_j \circ \Psi)}{\partial u_i}\frac{\partial}{\partial v_j}$$

Here each v_j in the partial derivatives $\frac{\partial v_j}{\partial u_i}$ are considered to be a locally defined function taking a point $p \in N$ to its v_j-coordinate.

For cotangent vectors (i.e. 1-forms), we talk about *pull-back* instead. According to Definition 3.14, Φ^* takes a cotangent vector in $T^*_{\Phi(p)}N$ to a cotangent vector in T^*_pM, defined as follows:

$$\Phi^*(dv^i)(X) = dv^i\,(\Phi_* X) \qquad \text{for any } X \in T_pM.$$

In terms of local coordinates, it is given by:

$$\Phi^*(dv^i) = \sum_j \frac{\partial(v_i \circ \Phi)}{\partial u_j}\,du^j.$$

The pull-back action by a smooth $\Phi : M \to N$ between manifolds can be extended to $(k,0)$-tensors (and hence to differential forms):

Definition 3.41 (Pull-Back on $(k,0)$-Tensors)**.** Let $\Phi : M \to N$ be a smooth map between two smooth manifolds. Given T a smooth $(k,0)$-tensor on N, then we define:

$$(\Phi^* T)_p (X_1, \ldots, X_k) = T_{\Phi(p)} (\Phi_*(X_1), \ldots, \Phi_*(X_k)) \quad \text{for any } X_1, \ldots, X_k \in T_p M$$

Remark 3.42. An equivalent way to state the definition is as follows: let $T_1, \ldots T_k \in TN$ be 1-forms on N, then we define:

$$\Phi^*(T_1 \otimes \cdots \otimes T_k) = (\Phi^* T_1) \otimes \cdots \otimes (\Phi^* T_k).$$

□

Remark 3.43. It is easy to verify that Φ^* is linear, in a sense that:

$$\Phi^*(aT + bS) = a\Phi^* T + b\Phi^* S$$

for any $(k,0)$-tensors T and S, and scalars a and b. □

Example 3.44. Let's start with an example on $\mathbb{R}^2$. Let $\Phi : \mathbb{R}^2 \to \mathbb{R}^3$ be a map defined by:

$$\Phi(x_1, x_2) = \left(e^{x_1+x_2}, \sin(x_1^2 x_2), x_1\right).$$

To avoid confusion, we use (x_1, x_2) to label the coordinates of the domain $\mathbb{R}^2$, and use (y_1, y_2, y_3) to denote the coordinates of the codomain $\mathbb{R}^3$. Then, we have:

$$\begin{aligned}
\Phi^*(dy^1)\left(\frac{\partial}{\partial x_1}\right) &= dy^1\left(\Phi_*\left(\frac{\partial}{\partial x_1}\right)\right) = dy^1\left(\frac{\partial \Phi}{\partial x_1}\right) \\
&= dy^1\left(\frac{\partial (y_1 \circ \Phi)}{\partial x_1}\frac{\partial}{\partial y_1} + \frac{\partial (y_2 \circ \Phi)}{\partial x_1}\frac{\partial}{\partial y_2} + \frac{\partial (y_3 \circ \Phi)}{\partial x_1}\frac{\partial}{\partial y_3}\right) \\
&= \frac{\partial (y_1 \circ \Phi)}{\partial x_1} = \frac{\partial}{\partial x_1} e^{x_1+x_2} = e^{x_1+x_2}.
\end{aligned}$$

Similarly, we have:

$$\Phi^*(dy^1)\left(\frac{\partial}{\partial x_2}\right) = \frac{\partial (y_1 \circ \Phi)}{\partial x_2} = \frac{\partial}{\partial x_2} e^{x_1+x_2} = e^{x_1+x_2}.$$

Therefore, $\Phi^*(dy^1) = e^{x_1+x_2} dx^1 + e^{x_1+x_2} dx^2 = e^{x_1+x_2}(dx^1 + dx^2)$. We leave it as an exercise for readers to verify that:

$$\begin{aligned}
\Phi^*(dy^2) &= 2x_1 x_2 \cos(x_1^2 x_2)\, dx^1 + x_1^2 \cos(x_1^2 x_2)\, dx^2 \\
\Phi^*(dy^3) &= dx^1
\end{aligned}$$

Let $f(y_1, y_2, y_3)$ be a scalar function on $\mathbb{R}^3$, and consider the $(2,0)$-tensor on $\mathbb{R}^3$:

$$T = f(y_1, y_2, y_3)\, dy^1 \otimes dy^2$$

The pull-back of T by Φ is given by:

$$\begin{aligned}
\Phi^* T &= f(y_1, y_2, y_3)\, \Phi^*(dy^1) \otimes \Phi^*(dy^2) \\
&= f(\Phi(x_1, x_2))\left(e^{x_1+x_2}(dx^1 + dx^2)\right) \otimes \left(2x_1 x_2 \cos(x_1^2 x_2)\, dx^1 + x_1^2 \cos(x_1^2 x_2)\, dx^2\right)
\end{aligned}$$

The purpose of writing $f(y_1, y_2, y_3)$ as $f(\Phi(x_1, x_2))$ is to leave the final expression in terms of functions and tensors in (x_1, x_2)-coordinates. □

Example 3.45. Let Σ be a regular surface in $\mathbb{R}^3$. The standard dot product in $\mathbb{R}^3$ is given by the following $(2,0)$-tensor:

$$\omega = dx \otimes dx + dy \otimes dy + dz \otimes dz.$$

Consider the inclusion map $\iota : \Sigma \to \mathbb{R}^3$. Although the input and output are the same under the map ι, the cotangents dx and $\iota^*(dx)$ are different! The former is a cotangent vector on $\mathbb{R}^3$, while $\iota^*(dx)$ is a cotangent vector on the surface Σ. If $(x, y, z) = \mathsf{F}(u, v)$ is a local parametrization of Σ, then $\iota^*(dx)$ should be in terms of du and dv, but not dx, dy and dz. Precisely, we have:

$$\begin{aligned}
\iota_*\left(\frac{\partial \mathsf{F}}{\partial u}\right) &= \frac{\partial \iota}{\partial u} := \frac{\partial(\iota \circ \mathsf{F})}{\partial u} = \frac{\partial \mathsf{F}}{\partial u} \\
\iota^*(dx)\left(\frac{\partial \mathsf{F}}{\partial u}\right) &= dx\left(\iota_*\left(\frac{\partial \mathsf{F}}{\partial u}\right)\right) = dx\left(\frac{\partial \mathsf{F}}{\partial u}\right) \\
&= dx\left(\frac{\partial x}{\partial u}\frac{\partial}{\partial x} + \frac{\partial y}{\partial u}\frac{\partial}{\partial y} + \frac{\partial z}{\partial u}\frac{\partial}{\partial z}\right) \\
&= \frac{\partial x}{\partial u}.
\end{aligned}$$

Similarly, we also have $\iota^*(dx)\left(\frac{\partial \mathsf{F}}{\partial v}\right) = \frac{\partial x}{\partial v}$, and hence:

$$\iota^*(dx) = \frac{\partial x}{\partial u}\,du + \frac{\partial x}{\partial v}\,dv.$$

As a result, we have:

$$\begin{aligned}
\iota^*\omega &= \iota^*(dx) \otimes \iota^*(dx) + \iota^*(dy) \otimes \iota^*(dy)\iota^*(dz) \otimes \iota^*(dz) \\
&= \left(\frac{\partial x}{\partial u}\,du + \frac{\partial x}{\partial v}\,dv\right) \otimes \left(\frac{\partial x}{\partial u}\,du + \frac{\partial x}{\partial v}\,dv\right) \\
&\quad + \left(\frac{\partial y}{\partial u}\,du + \frac{\partial y}{\partial v}\,dv\right) \otimes \left(\frac{\partial y}{\partial u}\,du + \frac{\partial y}{\partial v}\,dv\right) \\
&\quad + \left(\frac{\partial z}{\partial u}\,du + \frac{\partial z}{\partial v}\,dv\right) \otimes \left(\frac{\partial z}{\partial u}\,du + \frac{\partial z}{\partial v}\,dv\right).
\end{aligned}$$

After expansion and simplification, one will get:

$$\iota^*\omega = \frac{\partial \mathsf{F}}{\partial u} \cdot \frac{\partial \mathsf{F}}{\partial u}\,du \otimes du + \frac{\partial \mathsf{F}}{\partial u} \cdot \frac{\partial \mathsf{F}}{\partial v}\,du \otimes dv + \frac{\partial \mathsf{F}}{\partial v} \cdot \frac{\partial \mathsf{F}}{\partial u}\,dv \otimes du + \frac{\partial \mathsf{F}}{\partial v} \cdot \frac{\partial \mathsf{F}}{\partial v}\,dv \otimes dv,$$

which is the first fundamental form in Differential Geometry. □

Exercise 3.40. Let the unit sphere $\mathbb{S}^2$ be locally parametrized by spherical coordinates (θ, φ). Consider the $(2,0)$-tensor on $\mathbb{R}^3$:

$$\omega = x\,dy \otimes dz$$

Express the pull-back $\iota^*\omega$ in terms of (θ, φ).

One can derive a general formula (which you do not need to remember in practice) for the local expression of pull-backs. Consider local coordinates $\{u_i\}$ for M and $\{v_i\}$ for N, and write $(v_1, \ldots, v_n) = \Phi(u_1, \ldots, u_m)$ and

$$T = \sum_{i_1, \ldots, i_k = 1}^{n} T_{i_1 \cdots i_k}(v_1, \ldots, v_n)\,dv^{i_1} \otimes \cdots \otimes dv^{i_k}.$$

The pull-back $\Phi^* T$ then has the following local expression:

$$\begin{aligned}
(3.9)\quad \Phi^* T &= \sum_{i_1,\ldots,i_k=1}^{n} T_{i_1\cdots i_k}(v_1,\ldots,v_n)\,\Phi^*(dv^{i_1})\otimes\cdots\otimes\Phi^*(dv^{i_k})\\
&= \sum_{i_1,\ldots,i_k=1}^{n} T_{i_1\cdots i_k}(\Phi(u_1,\ldots,u_m))\left(\sum_{j_1=1}^{m}\frac{\partial v_{i_1}}{\partial u_{j_1}}\,du^{j_1}\right)\otimes\cdots\otimes\left(\sum_{j_k=1}^{m}\frac{\partial v_{i_k}}{\partial u_{j_k}}\,du^{j_k}\right)\\
&= \sum_{i_1,\ldots,i_k=1}^{n}\sum_{j_1,\ldots,j_k=1}^{m} T_{i_1\cdots i_k}(\Phi(u_1,\ldots,u_m))\frac{\partial v_{i_1}}{\partial u_{j_1}}\cdots\frac{\partial v_{i_k}}{\partial u_{j_k}}\,du^{j_1}\otimes\cdots\otimes du^{j_k}.
\end{aligned}$$

In view of $T_{i_1\cdots i_k}(v_1,\ldots,v_n) = T_{i_1\cdots i_k}(\Phi(u_1,\ldots,u_m))$ and the above local expression, we define

$$\Phi^* f := f\circ\Phi$$

for any scalar function of f. Using this notation, we then have $\Phi^*(fT) = (\Phi^* f)\,\Phi^* T$ for any scalar function f and $(k,0)$-tensor T.

Exercise 3.41. Let $\Phi : M \to N$ be a smooth map between smooth manifolds M and N, f be a smooth scalar function defined on N. Show that

$$\Phi^*(df) = d(\Phi^* f).$$

In particular, if $(v_1,\ldots,v_n)$ are local coordinates of N, we have $\Phi^*(dv^j) = d(\Phi^* v^j)$.

Example 3.46. Using the result from Exercise 3.41, one can compute the pull-back by inclusion map $\iota : \Sigma \to \mathbb{R}^3$ for regular surfaces Σ in $\mathbb{R}^3$. Suppose $\mathsf{F}(u,v)$ is a local parametrization of Σ, then:

$$\iota^*(dx) = d(\iota^* x) = d(x\circ\iota).$$

Although $x\circ\iota$ and x (as a coordinate function) have the same output, their domains are different! Namely, $x\circ\iota : \Sigma \to \mathbb{R}$ while $x : \mathbb{R}^3 \to \mathbb{R}$. Therefore, when computing $d(x\circ\iota)$, one should express it in terms of local coordinates (u,v) of Σ:

$$d(x\circ\iota) = \frac{\partial(x\circ\iota)}{\partial u}\,du + \frac{\partial(x\circ\iota)}{\partial v}\,dv = \frac{\partial x}{\partial u}\,du + \frac{\partial x}{\partial v}\,dv.$$

□

Recall that the tangent maps (i.e. push-forwards) acting on tangent vectors satisfy the chain rule: if $\Phi : M \to N$ and $\Psi : N \to P$ are smooth maps between smooth manifolds, then we have $(\Psi\circ\Phi)_* = \Psi_*\circ\Phi_*$. It is easy to extend the chain rule to $(k,0)$-tensors:

Theorem 3.47 (Chain Rule for $(k,0)$-tensors)**.** *Let $\Phi : M \to N$ and $\Psi : N \to P$ be smooth maps between smooth manifolds M, N and P, then the pull-back maps Φ^* and Ψ^* acting on $(k,0)$-tensors for any $k \geq 1$ satisfy the following chain rule:*

$$(\Psi\circ\Phi)^* = \Phi^*\circ\Psi^*. \tag{3.10}$$

Exercise 3.42. Prove Theorem 3.47.

Exercise 3.43. Denote id_M and id_{TM} to be the identity maps of a smooth manifold M and its tangent bundle respectively. Show that $(\mathrm{id}_M)^* = \mathrm{id}_{TM}$. Hence, show that if M and N are diffeomorphic, then for $k \geq 1$ the vector spaces of $(k,0)$-tensors $\otimes^k T^* M$ and $\otimes^k T^* N$ are isomorphic.

3.5.5. Pull-Back of Differential Forms. By linearity of the pull-back map, and the fact that differential forms are linear combinations of tensors, the pull-back map acts on differential forms by the following way:

$$\Phi^*(T_1 \wedge \cdots \wedge T_k) = \Phi^* T_1 \wedge \cdots \wedge \Phi^* T_k$$

for any 1-forms $T_1, \ldots, T_k$.

Example 3.48. Consider the map $\Phi : \mathbb{R}^2 \to \mathbb{R}^2$ given by:

$$\underbrace{\Phi(x_1, x_2)}_{(y_1, y_2)} = (x_1^2 - x_2, x_2^3).$$

By straight-forward computations, we have:

$$\begin{aligned} \Phi^*(dy^1) &= 2x_1 \, dx^1 - dx^2 \\ \Phi^*(dy^2) &= 3x_2 \, dx^2 \end{aligned}$$

Therefore, we have:

$$\Phi^*(dy^1 \wedge dy^2) = \Phi^*(dy^1) \wedge \Phi^*(dy^2) = 6x_1 x_2 \, dx^1 \wedge dx^2.$$

Note that $6x_1x_2$ is the Jacobian determinant $\det[\Phi_*]$. We will see soon that it is not a coincident, and it holds true in general. □

Although the computation of pull-back on differential forms is not much different from that on tensors, there are several distinctive features for pull-back on forms. One feature is that the pull-back on forms is closely related to Jacobian determinants:

Proposition 3.49. *Let $\Phi : M \to N$ be a smooth map between two smooth manifolds. Suppose $(u_1, \ldots, u_m)$ are local coordinates of M, and $(v_1, \ldots, v_n)$ are local coordinates of N, then for any $1 \le i_1, \ldots, i_k \le n$, we have:*

$$\Phi^*(dv^{i_1} \wedge \cdots \wedge dv^{i_k}) = \sum_{1 \le j_1 < \cdots < j_k \le m} \det \frac{\partial(v_{i_1}, \ldots, v_{i_k})}{\partial(u_{j_1}, \ldots, u_{j_k})} \, du^{j_1} \wedge \cdots \wedge du^{j_k}. \tag{3.11}$$

In particular, if $\dim M = \dim N = n$, *then we have:*

$$\Phi^*(dv^1 \wedge \cdots \wedge dv^n) = \det[\Phi_*] \, du^1 \wedge \cdots \wedge du^n \tag{3.12}$$

where $[\Phi_*]$ is the Jacobian matrix of Φ with respect to local coordinates $\{u_i\}$ and $\{v_i\}$, i.e.

$$[\Phi_*] = \frac{\partial(v_1, \ldots, v_n)}{\partial(u_1, \ldots, u_n)}.$$

Proof. Proceed as in the derivation of (3.9) by simply replacing all tensor products by wedge products, we get:

$$\begin{aligned} \Phi^*(dv^{i_1} \wedge \cdots \wedge dv^{i_k}) &= \sum_{j_1, \ldots, j_k = 1}^{m} \left(\frac{\partial v_{i_1}}{\partial u_{j_1}} \cdots \frac{\partial v_{i_k}}{\partial u_{j_k}} \, du^{j_1} \wedge \cdots \wedge du^{j_k} \right) \\ &= \sum_{\substack{j_1, \ldots, j_k = 1 \\ j_1, \ldots, j_k \text{ distinct}}}^{m} \left(\frac{\partial v_{i_1}}{\partial u_{j_1}} \cdots \frac{\partial v_{i_k}}{\partial u_{j_k}} \, du^{j_1} \wedge \cdots \wedge du^{j_k} \right) \end{aligned}$$

The second equality follows from the fact that $du^{j_1} \wedge \cdots \wedge du^{j_k} = 0$ if $\{j_1, \ldots, j_k\}$ are not all distinct. Each k-tuples $(j_1, \ldots, j_k)$ with distinct j_i's can be obtained by permuting a strictly increasing sequence of j's. Precisely, we have:

$$\begin{aligned}&\{(j_1,\ldots,j_k) : 1 \le j_1,\ldots,j_k \le n \text{ and } j_1,\ldots,j_k \text{ are all distinct}\}\\ &= \bigcup_{\sigma\in S_k} \{(j_{\sigma(1)},\ldots,j_{\sigma(k)}) : 1 \le j_1 < j_2 < \ldots < j_k \le n\}\end{aligned}$$

Therefore, we get:

$$\begin{aligned}&\Phi^*(dv^{i_1}\wedge\cdots\wedge dv^{i_k})\\ &= \sum_{1\le j_1<\ldots<j_k\le m}\sum_{\sigma\in S_k}\left(\frac{\partial v_{i_1}}{\partial u_{j_{\sigma(1)}}}\cdots\frac{\partial v_{i_k}}{\partial u_{j_{\sigma(k)}}}\,du^{j_{\sigma(1)}}\wedge\cdots\wedge du^{j_{\sigma(k)}}\right)\\ &= \sum_{1\le j_1<\ldots<j_k\le m}\sum_{\sigma\in S_k}\operatorname{sgn}(\sigma)\frac{\partial v_{i_1}}{\partial u_{j_{\sigma(1)}}}\cdots\frac{\partial v_{i_k}}{\partial u_{j_{\sigma(k)}}}\,du^{j_1}\wedge\cdots\wedge du^{j_k}\end{aligned}$$

By observing that $\sum_{\sigma\in S_k}\operatorname{sgn}(\sigma)\frac{\partial v_{i_1}}{\partial u_{j_{\sigma(1)}}}\cdots\frac{\partial v_{i_k}}{\partial u_{j_{\sigma(k)}}}$ is the determinant of $\left[\frac{\partial v_{i_p}}{\partial u_{j_q}}\right]_{1\le p,q\le k}$, the desired result (3.11) follows easily.

The second result (3.12) follows directly from (3.11). In case of $\dim M = \dim N = n$ and $k = n$, the only possible strictly increasing sequence $1 \le j_1 < \ldots < j_n \le n$ is $(j_1,\ldots,j_n) = (1,2,\ldots,n)$. □

Proposition 3.50. *Let $\Phi : M \to N$ be a smooth map between two smooth manifolds. For any $\omega \in \wedge^k T^*N$, we have:*

(3.13) $$\Phi^*(d\omega) = d(\Phi^*\omega).$$

To be precise, we say $\Phi^*(d_N\omega) = d_M(\Phi^*\omega)$, where $d_N : \wedge^k T^*N \to \wedge^{k+1}T^*N$ and $d_M : \wedge^k T^*M \to \wedge^{k+1}T^*M$ are the exterior derivatives on N and M respectively.

Proof. Let $\{u_j\}$ and $\{v_i\}$ be local coordinates of M and N respectively. By linearity, it suffices to prove (3.13) for the case $\omega = f\,dv^{i_1}\wedge\cdots\wedge dv^{i_k}$ where f is a locally defined scalar function. The proof follows from computing both LHS and RHS of (3.13):

$$\begin{aligned}d\omega &= df\wedge dv^{i_1}\wedge\cdots\wedge dv^{i_k}\\ \Phi^*(d\omega) &= \Phi^*(df)\wedge\Phi^*(dv^{i_1})\wedge\cdots\wedge\Phi^*(dv^{i_k})\\ &= d(\Phi^*f)\wedge d(\Phi^*v^{j_1})\wedge\cdots\wedge d(\Phi^*v^{j_k}).\end{aligned}$$

Here we have used Exercise 3.41. On the other hand, we have:

$$\begin{aligned}\Phi^*\omega &= (\Phi^*f)\,\Phi^*(dv^{j_1})\wedge\cdots\wedge\Phi^*(dv^{j_k})\\ &= (\Phi^*f)\,d(\Phi^*v^{i_1})\wedge\cdots\wedge d(\Phi^*v^{i_k})\\ d(\Phi^*\omega) &= d(\Phi^*f)\wedge d(\Phi^*v^{i_1})\wedge\cdots\wedge d(\Phi^*v^{i_k})\\ &\quad+\Phi^*f\,d\left(d(\Phi^*v^{i_1})\wedge\cdots\wedge d(\Phi^*v^{i_k})\right)\end{aligned}$$

Since $d^2 = 0$, each of $d(\Phi^*v^{i_q})$ is a closed 1-form. By Proposition 3.35 (product rule) and induction, we can conclude that:

$$d\left(d(\Phi^*v^{i_1})\wedge\cdots\wedge d(\Phi^*v^{i_k})\right) = 0$$

and so $d(\Phi^*\omega) = d(\Phi^*f)\wedge d(\Phi^*v^{i_1})\wedge\cdots\wedge d(\Phi^*v^{i_k})$ as desired. □

Exercise 3.44. Show that the pull-back of any closed form is closed, and the pull-back of any exact form is exact.

Exercise 3.45. Consider the unit sphere $\mathbb{S}^2$ locally parametrized by

$$\mathsf{F}(\theta, \varphi) = (\sin\varphi\cos\theta, \sin\varphi\sin\theta, \cos\varphi).$$

Define a map $\Phi : \mathbb{S}^2 \to \mathbb{R}^3$ by $\Phi(x, y, z) = (xz, yz, z^2)$, and consider a 2-form $\omega = z\,dx \wedge dy$. Compute $d\omega$, $\Phi^*(d\omega)$, $\Phi^*\omega$ and $d(\Phi^*\omega)$, and verify they satisfy Proposition 3.50.

3.5.6. Unification of Green's, Stokes' and Divergence Theorems. Given a submanifold M^m in $\mathbb{R}^n$, a differential form on $\mathbb{R}^n$ induces a differential form on M^m. For example, let C be a smooth regular curve in $\mathbb{R}^3$ parametrized by $\mathsf{r}(t) = (x(t), y(t), z(t))$. The 1-form:

$$\alpha = \alpha_x\,dx + \alpha_y\,dy + \alpha_z\,dz$$

is *a priori* defined on $\mathbb{R}^3$, but we can regard the coordinates (x, y, z) as functions on the curve C parametrized by $\mathsf{r}(t)$, then we have $dx = \dfrac{dx}{dt}\,dt$ and similarly for dy and dz. As such, dx can now be regarded as a 1-form on C. Therefore, the 1-form α on $\mathbb{R}^3$ induces a 1-form α (abuse in notation) on C:

$$\begin{aligned}\alpha &= \alpha_x(\mathsf{r}(t))\frac{dx}{dt}\,dt + \alpha_y(\mathsf{r}(t))\frac{dy}{dt}\,dt + \alpha_z(\mathsf{r}(t))\frac{dz}{dt}\,dt \\ &= \left(\alpha_x(\mathsf{r}(t))\frac{dx}{dt} + \alpha_y(\mathsf{r}(t))\frac{dy}{dt} + \alpha_z(\mathsf{r}(t))\frac{dz}{dt}\right)dt\end{aligned}$$

In practice, there is often no issue of using α to denote both the 1-form on $\mathbb{R}^3$ and its induced 1-form on C. To be (overly) rigorous over notations, we can use the inclusion map $\iota : C \to \mathbb{R}^3$ to distinguish them. The 1-form α on $\mathbb{R}^3$ is transformed into a 1-form $\iota^*\alpha$ on C by the pull-back of ι. From the previous subsection, we learned that:

$$\iota^*(dx) = d(\iota^* x) = d(x \circ \iota).$$

Note that dx and $d(x \circ \iota)$ are different in a sense that $x \circ \iota : C \to \mathbb{R}$ has the curve C as its domain, while $x : \mathbb{R}^3 \to \mathbb{R}$ has $\mathbb{R}^3$ as its domain. Therefore, we have:

$$d(x \circ \iota) = \frac{d(x \circ \iota)}{dt}\,dt = \frac{dx}{dt}\,dt.$$

In short, we may use $\iota^*(dx) = \dfrac{dx}{dt}\,dt$ to distinguish it from dx if necessary. Similarly, we may use $\iota^*\alpha$ to denote the induced 1-form of α on C:

$$\iota^*\alpha = \left(\alpha_x(\mathsf{r}(t))\frac{dx}{dt} + \alpha_y(\mathsf{r}(t))\frac{dy}{dt} + \alpha_z(\mathsf{r}(t))\frac{dz}{dt}\right)dt.$$

An induced 1-form on a curve in $\mathbb{R}^3$ is related to line integrals in Multivariable Calculus. Recall that the 1-form $\alpha = \alpha_x\,dx + \alpha_y\,dy + \alpha_z\,dz$ corresponds to the vector field $\mathsf{F} = \alpha_x\mathsf{i} + \alpha_y\mathsf{j} + \alpha_z\mathsf{k}$ on $\mathbb{R}^3$. In Multivariable Calculus, we denote $d\mathsf{r} = dx\mathsf{i} + dy\mathsf{j} + dz\mathsf{k}$ and

$$\mathsf{F} \cdot d\mathsf{r} = \left(\alpha_x\mathsf{i} + \alpha_y\mathsf{j} + \alpha_z\mathsf{k}\right) \cdot (dx\mathsf{i} + dy\mathsf{j} + dz\mathsf{k}) = \alpha.$$

The line integral $\displaystyle\int_C \mathsf{F} \cdot d\mathsf{r}$ over the curve $C \subset \mathbb{R}^3$ can be written using differential form notations:

$$\int_C \mathsf{F} \cdot d\mathsf{r} = \int_C \alpha \quad \text{or more rigorously:} \quad \int_C \iota^*\alpha.$$

Now consider a regular surface $M \subset \mathbb{R}^3$. Suppose $\mathsf{F}(u,v) = (x(u,v), y(u,v), z(u,v))$ is a smooth local parametrization of M. Consider a vector $\mathsf{G} = \beta_x \mathsf{i} + \beta_y \mathsf{j} + \beta_z \mathsf{k}$ on $\mathbb{R}^3$ and its corresponding 2-form on $\mathbb{R}^3$:

$$\beta = \beta_x \, dy \wedge dz + \beta_y \, dz \wedge dx + \beta_z \, dx \wedge dy.$$

Denote $\iota : M \to \mathbb{R}^3$ the inclusion map. The induced 2-form $\iota^*\beta$ on M is in fact related to the surface flux of G through M. Let's explain why:

$$\begin{aligned}
\iota^*(dy \wedge dz) &= (\iota^* dy) \wedge (\iota^* dz) = d(y \circ \iota) \wedge d(z \circ \iota) \\
&= \left(\frac{\partial y}{\partial u}\, du + \frac{\partial y}{\partial v}\, dv\right) \wedge \left(\frac{\partial z}{\partial u}\, du + \frac{\partial z}{\partial v}\, dv\right) \\
&= \left(\frac{\partial y}{\partial u}\frac{\partial z}{\partial v} - \frac{\partial z}{\partial u}\frac{\partial y}{\partial v}\right) du \wedge dv \\
&= \det \frac{\partial(y,z)}{\partial(u,v)}\, du \wedge dv.
\end{aligned}$$

Similarly, we have:

$$\begin{aligned}
\iota^*(dz \wedge dx) &= \det \frac{\partial(z,x)}{\partial(u,v)}\, du \wedge dv \\
\iota^*(dx \wedge dy) &= \det \frac{\partial(x,y)}{\partial(u,v)}\, du \wedge dv
\end{aligned}$$

All these show:

$$\iota^*\beta = \left(\beta_x \det \frac{\partial(y,z)}{\partial(u,v)} + \beta_y \det \frac{\partial(z,x)}{\partial(u,v)} + \beta_z \det \frac{\partial(x,y)}{\partial(u,v)}\right) du \wedge dv$$

Compared with the flux element $\mathsf{G} \cdot \mathsf{N}\, dS$ in Multivariable Calculus:

$$\begin{aligned}
\mathsf{G} \cdot \mathsf{N}\, dS &= \underbrace{\left(\beta_x \mathsf{i} + \beta_y \mathsf{j} + \beta_z \mathsf{k}\right)}_{\mathsf{G}} \cdot \underbrace{\frac{\frac{\partial \mathsf{F}}{\partial u} \times \frac{\partial \mathsf{F}}{\partial v}}{\left|\frac{\partial \mathsf{F}}{\partial u} \times \frac{\partial \mathsf{F}}{\partial v}\right|}}_{\mathsf{N}} \underbrace{\left|\frac{\partial \mathsf{F}}{\partial u} \times \frac{\partial \mathsf{F}}{\partial v}\right| dudv}_{dS} \\
&= \left(\beta_x \mathsf{i} + \beta_y \mathsf{j} + \beta_z \mathsf{k}\right) \cdot \left(\det \frac{\partial(y,z)}{\partial(u,v)}\mathsf{i} + \det \frac{\partial(z,x)}{\partial(u,v)}\mathsf{j} + \det \frac{\partial(x,y)}{\partial(u,v)}\mathsf{k}\right) \\
&= \left(\beta_x \det \frac{\partial(y,z)}{\partial(u,v)} + \beta_y \det \frac{\partial(z,x)}{\partial(u,v)} + \beta_z \det \frac{\partial(x,y)}{\partial(u,v)}\right) dudv,
\end{aligned}$$

the only difference is that $\iota^*\beta$ is in terms of the wedge product $du \wedge dv$ while the flux element $\mathsf{G} \cdot \mathsf{N}\, dS$ is in terms of $dudv$. Ignoring this minor difference (which will be addressed in the next chapter), the surface flux $\iint_M \mathsf{G} \cdot \mathsf{N}\, dS$ can be expressed in terms of differential forms in the following way:

$$\iint_M \mathsf{G} \cdot \mathsf{N}\, dS = \iint_M \beta \quad \text{or more rigorously:} \quad \iint_M \iota^*\beta$$

Recall that the classical Stokes' Theorem is related to line integrals of a curve and surface flux of a vector field. Based on the above discussion, we see that Stokes' Theorem can be restated in terms of differential forms. Consider the 1-form $\alpha = \alpha_x \, dx + \alpha_y \, dy + \alpha_z \, dz$ and its corresponding vector field $\mathsf{F} = \alpha_x \mathsf{i} + \alpha_y \mathsf{j} + \alpha_z \mathsf{k}$. We have already discussed that the 2-form $d\alpha$ corresponds to the vector field $\nabla \times \mathsf{F}$. Therefore, the surface flux of the vector field $\nabla \times \mathsf{F}$ through M can be expressed in terms of differential forms as:

$$\iint_M (\nabla \times \mathsf{F}) \cdot \mathsf{N}\, dS = \iint_M \iota^*(d\alpha) = \iint_M d(\iota^*\alpha).$$

If C is the boundary curve of M, then from our previous discussion we can write:

$$\int_C \mathsf{F} \cdot d\mathsf{r} = \int_C \iota^* \alpha.$$

The classical Stokes' Theorem asserts that:

$$\int_C \mathsf{F} \cdot d\mathsf{r} = \iint_M (\nabla \times \mathsf{F}) \cdot \mathsf{N} \, dS$$

which can be expressed in terms of differential form as:

$$\int_C \iota^* \alpha = \iint_M d(\iota^* \alpha) \quad \text{or simply:} \int_C \alpha = \iint_M d\alpha.$$

Due to this elegant way (although not very practical for physicists and engineers) of expressing Stokes' Theorem, we often denote the boundary of a surface M as ∂M, then the classical Stokes' Theorem can be expressed as:

$$\int_{\partial M} \alpha = \iint_M d\alpha.$$

Using differential forms, one can also express Divergence Theorem in Multivariable Calculus in a similar way as above. Let D be a solid region in $\mathbb{R}^3$ and ∂D be the boundary surface of D. Divergence Theorem in MATH 2023 asserts that:

$$\iint_{\partial D} \mathsf{G} \cdot \mathsf{N} \, dS = \iiint_D \nabla \cdot \mathsf{G} \, dV,$$

where $\mathsf{G} = \beta_x \mathsf{i} + \beta_y \mathsf{j} + \beta_z \mathsf{k}$. As discussed before, the LHS is $\iint_{\partial D} \beta$ where $\beta = \beta_x \, dy \wedge dz + \beta_y \, dz \wedge dx + \beta_z \, dx \wedge dy$. We have seen that:

$$d\beta = \left(\frac{\partial \beta_x}{\partial x} + \frac{\partial \beta_y}{\partial y} + \frac{\partial \beta_z}{\partial z} \right) dx \wedge dy \wedge dz,$$

which is (almost) the same as:

$$\nabla \cdot \mathsf{G} \, dV = \left(\frac{\partial \beta_x}{\partial x} + \frac{\partial \beta_y}{\partial y} + \frac{\partial \beta_z}{\partial z} \right) dxdydz.$$

Hence, the RHS of Divergence Theorem can be expressed as $\iiint_D d\beta$; and therefore we can rewrite Divergence Theorem as:

$$\iint_{\partial D} \beta = \iiint_D d\beta.$$

Again, the same expression! Stokes' and Divergence Theorems can therefore be unified. Green's Theorem can also be unified with Stokes' and Divergence Theorems as well. Try the exercise below:

Exercise 3.46. Let C be a simple closed smooth curve in $\mathbb{R}^2$ and R be the region enclosed by C in $\mathbb{R}^2$. Given a smooth vector field $\mathsf{F} = P\mathsf{i} + Q\mathsf{j}$ on $\mathbb{R}^2$, Green's Theorem asserts that:

$$\int_C \mathsf{F} \cdot d\mathsf{r} = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dxdy.$$

Express Green's Theorem using the language of differential forms.

3.5.7. Differential Forms and Maxwell's Equations. In Maxwell's theory of electromagnetism, the electric field $\mathsf{E} = E_1\mathsf{i} + E_2\mathsf{j} + E_3\mathsf{k}$ and the magnetic field $\mathsf{B} = B_1\mathsf{i} + B_2\mathsf{j} + B_3\mathsf{k}$ satisfy the following partial differential equations:

$$\nabla \times \mathsf{E} = -\frac{\partial \mathsf{B}}{\partial t} \qquad \text{(Faraday's Law)}$$
$$\nabla \cdot \mathsf{B} = 0 \qquad \text{(Gauss' Law for Magnetism)}$$

There are two more equations (namely Gauss' Law for Electricity and Ampére's Law) which we will not discuss here.

These two equations can be rewritten using differential forms in a very elegant way. First regard the E-field and B-field as differential forms on $\mathbb{R}^4$ with (x, y, z, t) as coordinates:

$$E = E_1\,dx + E_2\,dy + E_3\,dz$$
$$B = B_1\,dy \wedge dz + B_2\,dz \wedge dx + B_3\,dx \wedge dy$$

Note that E_i's and B_j's may depend on t although there is no dt above.

Exercise 3.47. Consider the 2-form $F := E \wedge dt + B$. Show that the Faraday's Law coupled with the Gauss' Law for Magnetism is equivalent to this single identity:

$$dF = 0$$

where d is the exterior derivative on $\mathbb{R}^4$.