Chapter 1

Regular Surfaces

"God made solids, but surfaces were the work of the devil."

Wolfgang Pauli

A manifold is a space which locally resembles an Euclidean space. Before we learn about manifolds in the next chapter, we first introduce the notion of regular surfaces in $\mathbb{R}^3$ which will give us some motivations of how a manifold is defined. As we will see later, many notions and concepts about manifolds are rooted from surfaces in $\mathbb{R}^3$.

1.1. Local Parametrizations

In Multivariable Calculus, we expressed a surface in $\mathbb{R}^3$ in two ways, namely using a parametrization $\mathsf{F}(u,v)$ or by a level set $f(x,y,z) = 0$. In this section, let us first focus on the former.

In MATH 2023, we used a parametrization $\mathsf{F}(u,v)$ to describe a surface in $\mathbb{R}^3$ and to calculate various geometric and physical quantities such as surface areas, surface integrals and surface flux. To start the course, we first look into several technical and analytical aspects concerning $\mathsf{F}(u,v)$, such as their domains and images, their differentiability, etc. In the past, we can usually cover (or almost cover) a surface by a single parametrization $\mathsf{F}(u,v)$. Take the unit sphere as an example. We learned that it can be parametrized with the help of spherical coordinates:

$$\mathsf{F}(\theta,\varphi) = (\sin\varphi\cos\theta, \sin\varphi\sin\theta, \cos\varphi)$$

where $0 < \theta < 2\pi$ and $0 < \varphi < \pi$. This parametrization covers almost every part of the sphere (except the north and south poles, and a half great circle connecting them). In order to cover the whole sphere, we need more parametrizations, such as $\mathsf{G}(\theta,\varphi) = (\sin\varphi\cos\theta, \sin\varphi\sin\theta, \cos\varphi)$ with domain $-\pi < \theta < \pi$ and $0 < \varphi < \pi$.

Since the image of either F or G does not cover the whole sphere (although almost), from now on we call them *local parametrizations*.

Definition 1.1 (Local Parametrizations of Class C^k)**.** Consider a subset $M \subset \mathbb{R}^3$. A function $\mathsf{F}(u,v) : \mathcal{U} \to \mathcal{O}$ from an open subset $\mathcal{U} \subset \mathbb{R}^2$ onto an open subset $\mathcal{O} \subset M$ is called a C^k *local parametrization* (or a C^k *local coordinate chart*) of M (where $k \geq 1$) if all of the following holds:

(1) $\mathsf{F} : \mathcal{U} \to \mathbb{R}^3$ is C^k when the codomain is regarded as $\mathbb{R}^3$.

(2) $\mathsf{F} : \mathcal{U} \to \mathcal{O}$ is a *homeomorphism*, meaning that $\mathsf{F} : \mathcal{U} \to \mathcal{O}$ is bijective, and both F and F^{-1} are continuous.

(3) For all $(u,v) \in \mathcal{U}$, the cross product:
$$\frac{\partial \mathsf{F}}{\partial u} \times \frac{\partial \mathsf{F}}{\partial v} \neq 0.$$

The coordinates (u,v) are called the *local coordinates* of M.

If $\mathsf{F} : \mathcal{U} \to M$ is of class C^k for any integer k, then F is said to be a C^∞ (or smooth) local parametrization.

Definition 1.2 (Surfaces of Class C^k)**.** A subset $M \subset \mathbb{R}^3$ is called a C^k *surface* in $\mathbb{R}^3$ if at every point $p \in M$, there exists an open subset $\mathcal{U} \subset \mathbb{R}^2$, an open subset $\mathcal{O} \subset M$ containing p and a C^k local parametrization $\mathsf{F} : \mathcal{U} \to \mathcal{O}$ which satisfies all three conditions stated in Definition 1.1.

We say M is a *regular surface* in $\mathbb{R}^3$ if it is a C^k surface for any integer k.

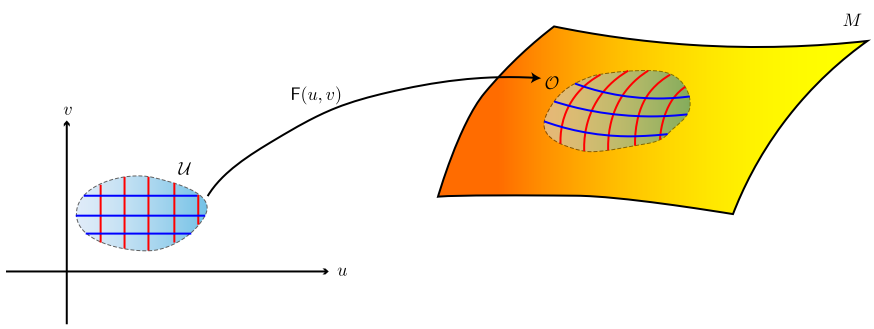
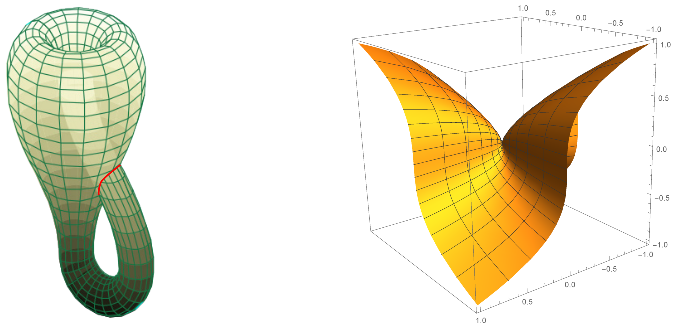


Figure 1.1. smooth local parametrization

To many students (myself included), the definition of regular surfaces looks obnoxious at the first glance. One way to make sense of it is to look at some examples and understand why each of the three conditions is needed in the definition.

The motivation behind condition (1) in the definition is that we are studying *differential* topology/geometry and so we want the parametrization to be differentiable as many times as we like. Condition (2) rules out surfaces that have self-intersection such as the Klein bottle (see Figure 1.2a). Finally, condition (3) guarantees the existence of a unique tangent plane at every point on M (see Figure 1.2b for a non-example).



(a) Klein Bottle has a self-intersection. **(b)** $\mathsf{F}(u,v) = (u^3, v^3, uv)$ fails condition (3).

Figure 1.2. Examples of non-smooth parametrizations

Example 1.3 (Graph of a Function)**.** Consider a smooth function $f(u,v) : \mathcal{U} \to \mathbb{R}$ defined on an open subset $\mathcal{U} \subset \mathbb{R}^2$. The graph of f, denoted by Γ_f, is a subset $\{(u,v,f(u,v)) : (u,v) \in \mathcal{U}\}$ of $\mathbb{R}^3$. One can parametrize Γ_f by a global parametrization:

$$\mathsf{F}(u,v) = (u,\ v,\ f(u,v)).$$

Condition (1) holds because f is given to be smooth. For condition (2), F is clearly one-to-one, and the image of F is the whole graph Γ_f. Regarding it as a map $\mathsf{F} : \mathcal{U} \to \Gamma_f$, the inverse map

$$\mathsf{F}^{-1}(x,y,z) = (x,y)$$

is clearly continuous. Therefore, $\mathsf{F} : \mathcal{U} \to \Gamma_f$ is a homeomorphism. To verify condition (3), we compute the cross product:

$$\frac{\partial \mathsf{F}}{\partial u} \times \frac{\partial \mathsf{F}}{\partial v} = \left(-\frac{\partial f}{\partial u},\ -\frac{\partial f}{\partial v},\ 1\right) \neq 0$$

for all $(u,v) \in \mathcal{U}$. Therefore, F is a smooth local parametrization of Γ_f. Since the image of this single smooth local parametrization covers all of Γ_f, we have proved that Γ_f is a regular surface. $\square$

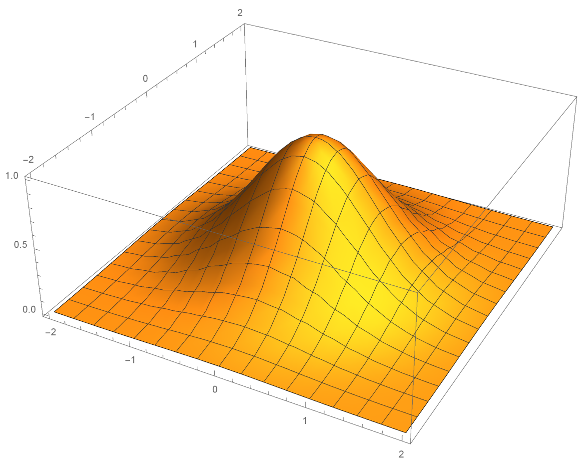


Figure 1.3. The graph of any smooth function is a regular surface.

Exercise 1.1. Show that $\mathsf{F}(u,v) : (0,2\pi) \times (0,1) \to \mathbb{R}^3$ defined by:

$$\mathsf{F}(u,v) = (\sin u, \sin 2u, v)$$

satisfies conditions (1) and (3) in Definition 1.1, but not condition (2). [Hint: Try to show F^{-1} is not continuous by finding a diverging sequence $\{(u_n, v_n)\}$ such that $\{\mathsf{F}(u_n, v_n)\}$ converges. See Figure 1.4 for reference.]

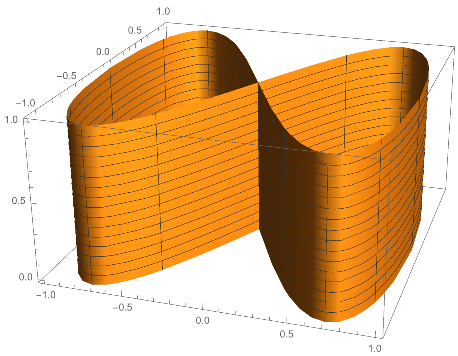


Figure 1.4. Plot of $\mathsf{F}(u,v)$ in Exercise 1.1

In Figure 1.3, one can observe that there are two families of curves on the surface. These curves are obtained by varying one of the (u,v)-variables while keeping the other constant. Precisely, they are the curves represented by $\mathsf{F}(u,v_0)$ and $\mathsf{F}(u_0,v)$ where u_0 and v_0 are fixed. As such, the partial derivatives $\frac{\partial \mathsf{F}}{\partial u}(p)$ and $\frac{\partial \mathsf{F}}{\partial v}(p)$ give a pair of tangent vectors on the surface at point p. Therefore, their cross product $\frac{\partial \mathsf{F}}{\partial u}(p) \times \frac{\partial \mathsf{F}}{\partial v}(p)$ is a normal vector to the surface at point p (see Figure 1.5). Here we have abused the notations for simplicity: $\frac{\partial \mathsf{F}}{\partial u}(p)$ means $\frac{\partial \mathsf{F}}{\partial u}$ evaluated at $(u,v) = \mathsf{F}^{-1}(p)$. Similarly for $\frac{\partial \mathsf{F}}{\partial v}(p)$.

Condition (3) requires that $\frac{\partial \mathsf{F}}{\partial u} \times \frac{\partial \mathsf{F}}{\partial v}$ is everywhere non-zero in the domain of F. An equivalent statement is that the vectors $\left\{\frac{\partial \mathsf{F}}{\partial u}(p), \frac{\partial \mathsf{F}}{\partial v}(p)\right\}$ are linearly independent for any $p \in \mathsf{F}(\mathcal{U})$.

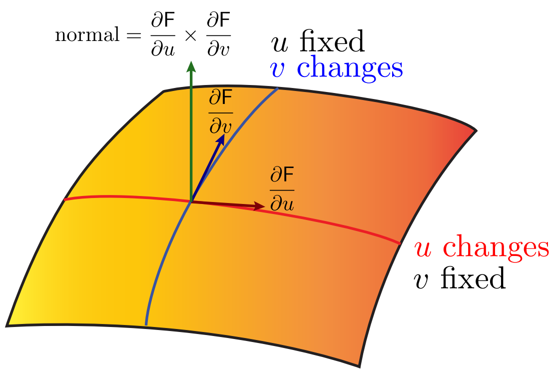


Figure 1.5. Tangent and normal vectors to a surface in $\mathbb{R}^3$

Example 1.4 (Sphere)**.** In $\mathbb{R}^3$, the unit sphere $\mathbb{S}^2$ centered at the origin can be represented by the equation $x^2 + y^2 + z^2 = 1$, or in other words, $z = \pm\sqrt{1 - x^2 - y^2}$. We can parametrize the upper and lower hemisphere by two separate local maps:

$$\mathsf{F}_1(u,v) = \left(u,\ v,\ \sqrt{1-u^2-v^2}\right) : B_1(0) \subset \mathbb{R}^2 \to \mathbb{S}^2_+$$

$$\mathsf{F}_2(u,v) = \left(u,\ v,\ -\sqrt{1-u^2-v^2}\right) : B_1(0) \subset \mathbb{R}^2 \to \mathbb{S}^2_-$$

where $B_1(0) = \{(u,v) : u^2 + v^2 < 1\}$ is the *open* unit disk in $\mathbb{R}^2$ centered at the origin, and $\mathbb{S}^2_+$ and $\mathbb{S}^2_-$ are the upper and lower hemispheres of $\mathbb{S}^2$ respectively. Since $B_1(0)$ is open, the functions $\pm\sqrt{1-u^2-v^2}$ are smooth and so according to the previous example, both F_1 and F_2 are smooth local parametrizations.

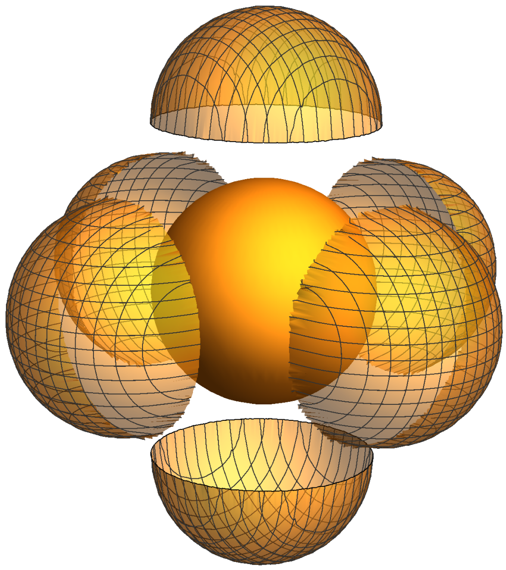


Figure 1.6. A unit sphere covered by six parametrization charts

However, not all points on the sphere are covered by $\mathbb{S}^2_+$ and $\mathbb{S}^2_-$, since points on the equator are not. In order for show that $\mathbb{S}^2$ is a regular surface, we need to write down more smooth local parametrization(s) so that each point on the sphere can be covered by at least one smooth local parametrization chart. One can construct four more smooth local parametrizations (left, right, front and back) similar to F_1 and F_2 (see Figure 1.6). It is left as an exercise for readers to write down the other four parametrizations. These six parametrizations are all smooth and they cover the whole sphere. Therefore, it shows the sphere is a regular surface. □

Exercise 1.2. Write down the left, right, front and back parametrizations F_i's $(i = 3, 4, 5, 6)$ of the sphere as shown in Figure 1.6. Indicate clearly the domain and range of each F_i.

Example 1.5 (Sphere: Revisited)**.** We can in fact cover the sphere by two smooth local parametrization described below. Define $\mathsf{F}_+(u,v) : \mathbb{R}^2 \to \mathbb{S}^2 \backslash \{(0,0,1)\}$ where:

$$\mathsf{F}_+(u,v) = \left(\frac{2u}{u^2+v^2+1}, \frac{2v}{u^2+v^2+1}, \frac{u^2+v^2-1}{u^2+v^2+1} \right)$$

It is called the *stereographic parametrization* of the sphere (see Figure 1.7) , which assigns each point $(u,v,0)$ on the xy-plane of $\mathbb{R}^3$ to a point where the line segment joining $(u,v,0)$ and the north pole $(0,0,1)$ intersects the sphere. Clearly F_+ is a smooth function. We leave it as exercise for readers to verify that F_+ satisfies condition (3) and that $\mathsf{F}_+^{-1} : \mathbb{S}^2 \backslash \{(0,0,1)\} \to \mathbb{R}^2$ is given by:

$$\mathsf{F}_+^{-1}(x,y,z) = \left(\frac{x}{1-z}, \frac{y}{1-z} \right).$$

As $z \neq 1$ for every (x,y,z) in the domain of F_+^{-1}, it is a continuous function. Therefore, F_+ is a smooth local parametrization. The inverse map F_+^{-1} is commonly called the stereographic projection of the sphere.

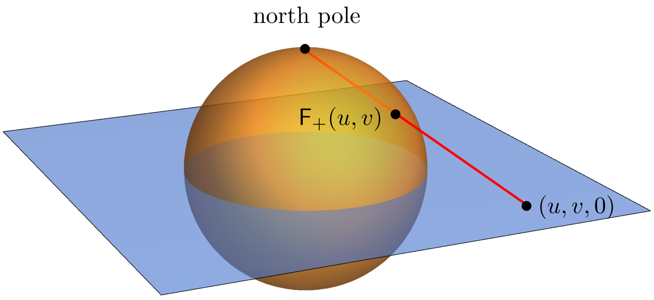


Figure 1.7. Stereographic parametrization of the sphere

Note that the range of F_+ does not include the point $(0,0,1)$. In order to show that the sphere is a regular surface, we need to cover it by another parametrization $\mathsf{F}_- : \mathbb{R}^2 \to \mathbb{S} \backslash \{(0,0,-1)\}$ which assigns each point $(u,v,0)$ on the xy-plane to a point where the line segment joining $(u,v,0)$ and the south pole $(0,0,-1)$ intersects the sphere. It is an exercise for readers to write down the explicit parametrization F_-. □

Exercise 1.3. Verify that F_+ in Example 1.4 satisfies condition (3) in Definition 1.1, and that the inverse map $\mathsf{F}_+^{-1} : \mathbb{S}^2 \backslash \{(0,0,1)\} \to \mathbb{R}^2$ is given as stated. [Hint: Write down $\mathsf{F}_+(u,v) = (x,y,z)$ and solve (u,v) in terms of (x,y,z). Begin by finding u^2+v^2 in terms of z.]

Furthermore, write down explicitly the map F_- described in Example 1.4, and find its inverse map F_-^{-1}.

Exercise 1.4. Find smooth local parametrizations which together cover the whole ellipsoid:

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

where a, b and c are positive constants.

Exercise 1.5. Let M be the cylinder $\{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 = 1\}$. The purpose of this exercise is to construct a smooth local parametrization analogous to the stereographic parametrization in Example 1.4:

Consider the unit circle $x^2 + y^2 = 1$ on the xy-plane. For each point $(u, 0)$ on the x-axis, we construct a straight-line joining the point $(0, 1)$ and $(u, 0)$. This line intersects the unit circle at a unique point p. Denote the xy-coordinates of p by $(x(u), y(u))$.

(a) Find the coordinates $(x(u), y(u))$ in terms of u.

(b) Define:
$$\mathsf{F}_1(u, v) = (x(u), y(u), v)$$
with $\mathbb{R}^2$ as its domain. Describe the image of F_1.

(c) Denote $\mathcal{O}_1$ to be the image of F_1. Verify that $\mathsf{F}_1 : \mathbb{R}^2 \to \mathcal{O}_1$ is smooth local parametrization of M.

(d) Construct another smooth local parametrization F_2 such that the images of F_1 and F_2 cover the whole surface M (hence establish that M is a regular surface).

Let's also look at a non-example of smooth local parametrizations. Consider the map:
$$\mathsf{G}(u, v) = (u^3, v^3, 0), \quad (u, v) \in \mathbb{R} \times \mathbb{R}.$$
It is a smooth, injective map from $\mathbb{R}^2$ onto the xy-plane Π of $\mathbb{R}^3$, i.e. $\mathsf{G} : \mathbb{R}^2 \to \Pi$. However, it can be computed that
$$\frac{\partial \mathsf{G}}{\partial u}(0, 0) = \frac{\partial \mathsf{G}}{\partial v}(0, 0) = \mathsf{0}$$
and so condition (3) in Definition 1.1 does not hold. The map G is not a smooth local parametrization of Π. However, note that Π is a regular surface because $\mathsf{F}(u, v) = (u, v, 0)$ is a smooth global parametrization of Π, even though G is not a "good" parametrization.

In order to show M is a regular surface, what we need is to show at every point $p \in M$ there is *at least one* smooth local parametrization F near p. However, to show that M is not a regular surface, one then needs to come up with a point $p \in M$ such that there is *no* smooth local parametrization near that point p (which may not be easy).

1.2. Level Surfaces

Many surfaces are defined using an equation such as $x^2+y^2+z^2=1$, or $x^2+y^2=z^2+1$. They are level sets of a function $g(x,y,z)$. In this section, we are going to prove a theorem that allows us to show easily that some level sets $g^{-1}(c)$ are regular surfaces.

Theorem 1.6. *Let $g(x,y,z):\mathbb{R}^3\to\mathbb{R}$ be a smooth function of three variables. Consider a non-empty level set $g^{-1}(c)$ where c is a constant. If $\nabla g(x_0,y_0,z_0)\neq 0$ at all points $(x_0,y_0,z_0)\in g^{-1}(c)$, then the level set $g^{-1}(c)$ is a regular surface.*

Proof. The key idea of the proof is to use the Implicit Function Theorem. Given any point $p=(x_0,y_0,z_0)\in g^{-1}(c)$, since $\nabla g(x_0,y_0,z_0)\neq(0,0,0)$, at least one of the first partials:

$$\frac{\partial g}{\partial x}(p),\ \frac{\partial g}{\partial y}(p),\ \frac{\partial g}{\partial z}(p)$$

is non-zero. Without loss of generality, assume $\frac{\partial g}{\partial z}(p)\neq 0$, then the Implicit Function Theorem shows that locally around the point p, the level set $g^{-1}(c)$ can be regarded as a graph $z=f(x,y)$ of some smooth function f of (x,y). To be precise, there exists an open set $\mathcal{O}$ of $g^{-1}(c)$ containing p such that there is a smooth function $f(x,y):\mathcal{U}\subset\mathbb{R}^2\to\mathbb{R}$ from an open set $\mathcal{U}$ such that $(x,y,f(x,y))\in\mathcal{O}\subset g^{-1}(c)$ for any $(x,y)\in\mathcal{U}$. As such, the smooth local parametrization $\mathsf{F}:\mathcal{U}\to\mathcal{O}$ defined by:

$$\mathsf{F}(u,v)=(u,v,f(u,v))$$

is a smooth local parametrization of $g^{-1}(c)$.

In the case where $\frac{\partial g}{\partial y}(p)\neq 0$, the above argument is similar as locally around p one can regard $g^{-1}(c)$ as a graph $y=h(x,z)$ for some smooth function h. Similar in the case $\frac{\partial g}{\partial x}(p)\neq 0$.

Since every point p can be covered by the image of a smooth local parametrization, the level set $g^{-1}(c)$ is a regular surface. □

Example 1.7. The unit sphere $x^2+y^2+z^2=1$ is a level surface $g^{-1}(1)$ where $g(x,y,z):=x^2+y^2+z^2$. The gradient vector $\nabla g=(2x,\ 2y,\ 2z)$ is zero only when $(x,y,z)=(0,0,0)$. Since the origin is not on the unit sphere, we have $\nabla g(x_0,y_0,z_0)\neq(0,0,0)$ for any $(x_0,y_0,z_0)\in g^{-1}(1)$. Therefore, the unit sphere is a regular surface.

Similarly, one can also check that the surface $x^2+y^2=z^2+1$ is a regular surface. It is a level set $h^{-1}(1)$ where $h(x,y,z)=x^2+y^2-z^2$. Since $\nabla h=(2x,\ 2y,\ -2z)$, the origin is the only point p at which $\nabla h(p)=(0,0,0)$ and it is not on the level set $h^{-1}(1)$. Therefore, $h^{-1}(1)$ is a regular surface. □

However, the cone $x^2+y^2=z^2$ cannot be shown to be a regular surface using Theorem 1.6. It is a level surface $h^{-1}(0)$ where $h(x,y,z):=x^2+y^2-z^2$. The origin $(0,0,0)$ is on the cone and $\nabla h(0,0,0)=(0,0,0)$. Theorem 1.6 fails to give any conclusion.

The converse of Theorem 1.6 is not true. Consider $g(x,y,z)=z^2$, then $g^{-1}(0)$ is the xy-plane which is clearly a regular surface. However, $\nabla g=(0,0,2z)$ is zero at the origin which is contained in the xy-plane.

Exercise 1.6. [**dC76**, P.66] Let $f(x,y,z) = (x+y+z-1)^2$. For what values of c is the set $f^{-1}(c)$ a regular surface?

Exercise 1.7. A torus is defined by the equation:

$$z^2 = R^2 - \left(\sqrt{x^2+y^2} - r\right)^2$$

where $R > r > 0$ are constants. Show that it is a regular surface.

The proof of Theorem 1.6 makes use of the Implicit Function Theorem which is an existence result. It shows a certain level set is a regular surface, but it fails to give an explicit smooth local parametrization around each point.

There is one practical use of Theorem 1.6 though. Suppose we are given $\mathsf{F}(u,v)$ which satisfies conditions (1) and (3) in Definition 1.1 and that F is continuous and F^{-1} exists. In order to verify that it is a smooth local parametrization, we need to prove continuity of F^{-1}, which is sometimes difficult. Here is one example:

$$\mathsf{F}(u,v) = (\sin u \, \cos v, \, \sin u \, \sin v, \, \cos u), \quad 0 < u < \pi, \, 0 < v < 2\pi$$

is a smooth local parametrization of a unit sphere. It is clearly a smooth map from $(0,\pi) \times (0,2\pi) \subset \mathbb{R}^2$ to $\mathbb{R}^3$, and it is quite straight-forward to verify condition (3) in Definition 1.1 and that F is one-to-one. However, it is rather difficult to write down an explicit F^{-1}, let alone to show it is continuous.

The following result tells us that if the surface is given by a level set satisfying conditions stated in Theorem 1.6, and F satisfies conditions (1) and (3), then F^{-1} is *automatically* continuous. Precisely, we have the following:

Proposition 1.8. *Assume all given conditions stated in Theorem 1.6. Furthermore, suppose* $\mathsf{F}(u,v)$ *is a bijective map from an open set* $\mathcal{U} \subset \mathbb{R}^2$ *to an open set* $\mathcal{O} \subset M := g^{-1}(c)$ *which satisfies conditions (1) and (3) in Definition 1.1. Then,* F *satisfies condition (2) as well and hence is a smooth local parametrization of* $g^{-1}(c)$.

Proof. Given any point $p \in g^{-1}(c)$, we can assume without loss of generality that $\frac{\partial g}{\partial z}(p) \neq 0$. Recall from Multivariable Calculus that $\nabla g(p)$ is a normal vector to the level surface $g^{-1}(c)$ at point p. Furthermore, if $\mathsf{F}(u,v)$ is a map satisfying conditions (1) and (3) of Definition 1.1, then $\frac{\partial \mathsf{F}}{\partial u}(p) \times \frac{\partial \mathsf{F}}{\partial v}(p)$ is also a normal vector to $g^{-1}(c)$ at p.

Now that the k-component of $\nabla g(p)$ is non-zero since $\frac{\partial g}{\partial z}(p) \neq 0$, so the k-component of the cross product $\frac{\partial \mathsf{F}}{\partial u}(p) \times \frac{\partial \mathsf{F}}{\partial v}(p)$ is also non-zero. If we express $\mathsf{F}(u,v)$ as:

$$\mathsf{F}(u,v) = (x(u,v), y(u,v), z(u,v)),$$

then the k-component of $\frac{\partial \mathsf{F}}{\partial u}(p) \times \frac{\partial \mathsf{F}}{\partial v}(p)$ is given by:

$$\left(\frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial y}{\partial u}\frac{\partial x}{\partial v}\right)(p) = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}(p).$$

Define $\pi : \mathbb{R}^3 \to \mathbb{R}^2$ by $\pi(x,y,z) = (x,y)$. The above shows that the composition $\pi \circ \mathsf{F}$ given by

$$(\pi \circ \mathsf{F})(u,v) = (x(u,v), y(u,v))$$

has non-zero Jacobian determinant at p. By the Inverse Function Theorem, $\pi \circ \mathsf{F}$ has a smooth local inverse near p. In particular, $(\pi \circ \mathsf{F})^{-1}$ is continuous near p.

Finally, by the fact that $(\pi \circ \mathsf{F}) \circ \mathsf{F}^{-1} = \pi$ and that $(\pi \circ \mathsf{F})^{-1}$ exists and is continuous locally around p, we can argue that $\mathsf{F}^{-1} = (\pi \circ \mathsf{F})^{-1} \circ \pi$ is also continuous near p. It completes the proof. $\square$

Exercise 1.8. Rewrite the proof of Proposition 1.8 by assuming $\frac{\partial g}{\partial y}(p) \neq 0$ instead.

Example 1.9. We have already shown that the unit sphere $x^2 + y^2 + z^2 = 1$ is a regular surface using Theorem 1.6 by regarding it is the level set $g^{-1}(1)$ where $g(x,y,z) = x^2 + y^2 + z^2$. We also discussed that

$$\mathsf{F}(u,v) = (\sin u \, \cos v, \, \sin u \, \sin v, \, \cos u), \quad 0 < u < \pi, \, 0 < v < 2\pi$$

is a *possible* smooth local parametrization. It is clearly smooth, and by direct computation, one can show

$$\frac{\partial \mathsf{F}}{\partial u} \times \frac{\partial \mathsf{F}}{\partial v} = \sin u \, (\sin u \cos v, \, \sin u \sin v, \, \cos u)$$

and so $\left| \frac{\partial \mathsf{F}}{\partial u} \times \frac{\partial \mathsf{F}}{\partial v} \right| = \sin u \neq 0$ for any (u,v) in the domain $(0,\pi) \times (0,2\pi)$. We leave it as an exercise for readers to verify that F is one-to-one (and so bijective when its codomain is taken to be its image).

Condition (2) is not easy to verify because it is difficult to write down the inverse map F^{-1} explicitly. However, thanks for Proposition 1.8, F is a smooth local parametrization since it satisfies conditions (1) and (3), and it is one-to-one. $\square$

Exercise 1.9. Consider that the *Mercator projection* of the unit sphere:

$$\mathsf{F}(u,v) = \left(\frac{\cos v}{\cosh u}, \frac{\sin v}{\cosh u}, \frac{\sinh u}{\cosh u} \right)$$

where $\sinh u := \frac{1}{2}(e^u - e^{-u})$ and $\cosh u := \frac{1}{2}(e^u + e^{-u})$.

(a) What are the domain and range of F?

(b) Show that F is a smooth local parametrization.

Exercise 1.10. Consider the following parametrization of a torus $\mathbb{T}^2$:

$$\mathsf{F}(u,v) = ((R\cos u + r)\cos v, \, (R\cos u + r)\sin v, \, R\sin u)$$

where $(u,v) \in (0,2\pi) \times (0,2\pi)$, and $R > r > 0$ are constants. Show that F is a smooth local parametrization.

1.3. Transition Maps

Let $M \subset \mathbb{R}^3$ be a regular surface, and $\mathsf{F}_\alpha(u_1, u_2) : \mathcal{U}_\alpha \to M$ and $\mathsf{F}_\beta(v_1, v_2) : \mathcal{U}_\beta \to M$ be two smooth local parametrizations of M with overlapping images, i.e. $\mathcal{W} := \mathsf{F}_\alpha(\mathcal{U}_\alpha) \cap \mathsf{F}_\beta(\mathcal{U}_\beta) \neq \emptyset$. Under this set-up, it makes sense to define the maps $\mathsf{F}_\beta^{-1} \circ \mathsf{F}_\alpha$ and $\mathsf{F}_\alpha^{-1} \circ \mathsf{F}_\beta$. However, we need to shrink their domains so as to guarantee they are well-defined. Precisely:

$$(\mathsf{F}_\beta^{-1} \circ \mathsf{F}_\alpha) : \mathsf{F}_\alpha^{-1}(\mathcal{W}) \to \mathsf{F}_\beta^{-1}(\mathcal{W})$$
$$(\mathsf{F}_\alpha^{-1} \circ \mathsf{F}_\beta) : \mathsf{F}_\beta^{-1}(\mathcal{W}) \to \mathsf{F}_\alpha^{-1}(\mathcal{W})$$

Note that $\mathsf{F}_\alpha^{-1}(\mathcal{W})$ and $\mathsf{F}_\beta^{-1}(\mathcal{W})$ are open subsets of $\mathcal{U}_\alpha$ and $\mathcal{U}_\beta$ respectively. The map $\mathsf{F}_\beta^{-1} \circ \mathsf{F}_\alpha$ describes a relation between two sets of coordinates (u_1, u_2) and (v_1, v_2) of M. In other words, one can regard $\mathsf{F}_\beta^{-1} \circ \mathsf{F}_\alpha$ as a *change-of-coordinates*, or *transition* map and we can write:

$$\mathsf{F}_\beta^{-1} \circ \mathsf{F}_\alpha(u_1, u_2) = (v_1(u_1, u_2),\, v_2(u_1, u_2)).$$

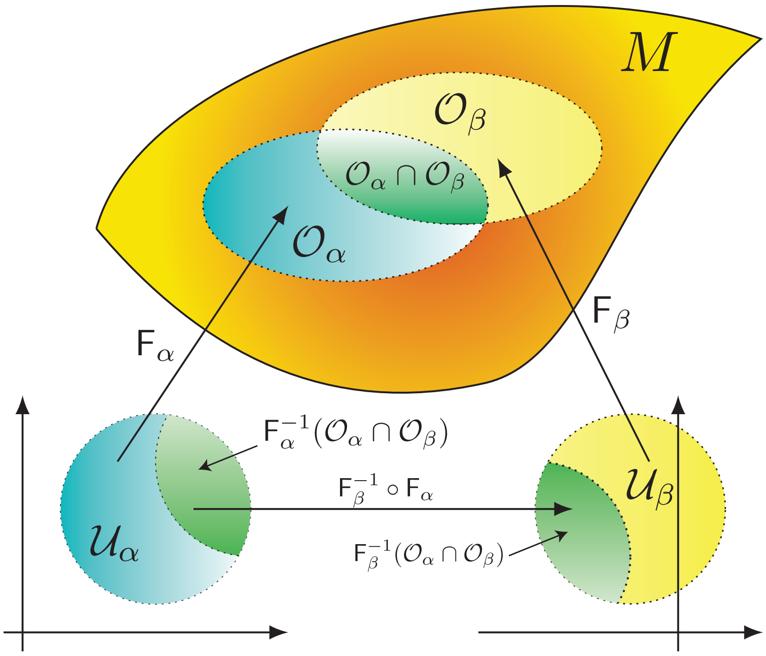


Figure 1.8. Transition maps

One goal of this section is to show that this transition map $\mathsf{F}_\beta^{-1} \circ \mathsf{F}_\alpha$ is smooth provided that F_α and F_β are two overlapping smooth local parametrizations. Before we present the proof, let us look at some examples of transition maps.

Example 1.10. The xy-plane Π in $\mathbb{R}^3$ is a regular surface which admits a global smooth parametrization $\mathsf{F}_\alpha(x,y) = (x,y,0) : \mathbb{R}^2 \to \Pi$. Another way to locally parametrize Π is by polar coordinates $\mathsf{F}_\beta : (0,\infty) \times (0,2\pi) \to \Pi$

$$\mathsf{F}_\beta(r,\theta) = (r\cos\theta, r\sin\theta, 0)$$

Readers should verify that they are smooth local parametrizations. The image of F_α is the entire xy-plane Π, whereas the image of F_β is the xy-plane with the origin and positive x-axis removed. The transition map $\mathsf{F}_\alpha^{-1} \circ \mathsf{F}_\beta$ is given by:

$$\begin{aligned}\mathsf{F}_\alpha^{-1} \circ \mathsf{F}_\beta : (0,\infty) \times (0,2\pi) &\to \mathbb{R}^2 \backslash \{(x,0) : x \geq 0\} \\ (r,\theta) &\mapsto (r\cos\theta, r\sin\theta)\end{aligned}$$

To put it in a simpler form, we can say $(x(r,\theta), y(r,\theta)) = (r\cos\theta, r\sin\theta)$. □

Exercise 1.11. Consider the stereographic parametrizations F_+ and F_- in Example 1.5. Compute the transition maps $\mathsf{F}_+^{-1} \circ \mathsf{F}_-$ and $\mathsf{F}_-^{-1} \circ \mathsf{F}_+$. State the maximum possible domain for each map. Are they smooth on their domains?

Exercise 1.12. The unit cylinder Σ^2 in $\mathbb{R}^3$ can be covered by two local parametrizations:

$$\begin{aligned}\mathsf{F} : (0,2\pi) \times \mathbb{R} &\to \Sigma^2 & \widetilde{\mathsf{F}} : (-\pi,\pi) \times \mathbb{R} &\to \Sigma^2 \\ \mathsf{F}(\theta,z) &:= (\cos\theta, \sin\theta, z) & \widetilde{\mathsf{F}}(\widetilde{\theta},\widetilde{z}) &:= (\cos\widetilde{\theta}, \sin\widetilde{\theta}, \widetilde{z})\end{aligned}$$

Compute the transition maps $\mathsf{F}^{-1} \circ \widetilde{\mathsf{F}}$ and $\widetilde{\mathsf{F}}^{-1} \circ \mathsf{F}$. State their maximum possible domains. Are they smooth on their domains?

Exercise 1.13. The Möbius strip Σ^2 in $\mathbb{R}^3$ can be covered by two local parametrizations:

$$\mathsf{F} : (-1,1) \times (0,2\pi) \to \Sigma^2 \qquad \widetilde{\mathsf{F}} : (-1,1) \times (-\pi,\pi) \to \Sigma^2$$

$$\mathsf{F}(u,\theta) = \begin{bmatrix} \left(3 + u\cos\frac{\theta}{2}\right)\cos\theta \\ \left(3 + u\cos\frac{\theta}{2}\right)\sin\theta \\ u\sin\frac{\theta}{2} \end{bmatrix} \qquad \widetilde{\mathsf{F}}(\widetilde{u},\widetilde{\theta}) = \begin{bmatrix} \left(3 + \widetilde{u}\cos\frac{\widetilde{\theta}}{2}\right)\cos\widetilde{\theta} \\ \left(3 + \widetilde{u}\cos\frac{\widetilde{\theta}}{2}\right)\sin\widetilde{\theta} \\ \widetilde{u}\sin\frac{\widetilde{\theta}}{2} \end{bmatrix}$$

Compute the transition maps, state their maximum possible domains and verify that they are smooth.

The proposition below shows that the transition maps between any pair of smooth local parametrizations are smooth:

Proposition 1.11. *Let $M \subset \mathbb{R}^3$ be a regular surface, and $\mathsf{F}_\alpha(u_1,u_2) : \mathcal{U}_\alpha \to M$ and $\mathsf{F}_\beta(v_1,v_2) : \mathcal{U}_\beta \to M$ be two smooth local parametrizations of M with overlapping images, i.e. $\mathcal{W} := \mathsf{F}_\alpha(\mathcal{U}_\alpha) \cap \mathsf{F}_\beta(\mathcal{U}_\beta) \neq \emptyset$. Then, the transition maps defined below are also smooth maps:*

$$(\mathsf{F}_\beta^{-1} \circ \mathsf{F}_\alpha) : \mathsf{F}_\alpha^{-1}(\mathcal{W}) \to \mathsf{F}_\beta^{-1}(\mathcal{W})$$
$$(\mathsf{F}_\alpha^{-1} \circ \mathsf{F}_\beta) : \mathsf{F}_\beta^{-1}(\mathcal{W}) \to \mathsf{F}_\alpha^{-1}(\mathcal{W})$$

Proof. It suffices to show $\mathsf{F}_\beta^{-1} \circ \mathsf{F}_\alpha$ is smooth as the other one $\mathsf{F}_\alpha^{-1} \circ \mathsf{F}_\beta$ can be shown by symmetry. Furthermore, since differentiability is a local property, we may fix a point $p \in \mathcal{W} \subset M$ and show that $\mathsf{F}_\beta^{-1} \circ \mathsf{F}_\alpha$ is smooth at the point $\mathsf{F}_\alpha^{-1}(p)$.

By condition of (3) of smooth local parametrizations, we have:

$$\frac{\partial \mathsf{F}_\alpha}{\partial u_1}(p) \times \frac{\partial \mathsf{F}_\alpha}{\partial u_2}(p) \neq 0$$

By straight-forward computations, one can show that this cross product is given by:

$$\frac{\partial \mathsf{F}_\alpha}{\partial u_1} \times \frac{\partial \mathsf{F}_\alpha}{\partial u_2} = \left(\det \frac{\partial(y,z)}{\partial(u_1,u_2)}(p),\ \det \frac{\partial(z,x)}{\partial(u_1,u_2)}(p),\ \det \frac{\partial(x,y)}{\partial(u_1,u_2)}(p) \right).$$

Hence, at least one of the determinants is non-zero. Without loss of generality, assume that:

$$\det \frac{\partial(x,y)}{\partial(u_1,u_2)}(p) \neq 0.$$

Both $\frac{\partial \mathsf{F}_\alpha}{\partial u_1}(p) \times \frac{\partial \mathsf{F}_\alpha}{\partial u_2}(p)$ and $\frac{\partial \mathsf{F}_\beta}{\partial v_1}(p) \times \frac{\partial \mathsf{F}_\beta}{\partial v_2}(p)$ are normal vectors to the surface at p. Given that the former has non-zero k-component, then so does the latter. Therefore, we have:

$$\det \frac{\partial(x,y)}{\partial(v_1,v_2)}(p) \neq 0.$$

Then we proceed as in the proof of Proposition 1.8. Define $\pi(x,y,z) = (x,y)$, then

$$\begin{aligned} \pi \circ \mathsf{F}_\beta : \mathcal{U}_\beta &\to \mathbb{R}^2 \\ (v_1, v_2) &\mapsto (x(v_1,v_2), y(v_1,v_2)) \end{aligned}$$

has non-zero Jacobian determinant $\det \frac{\partial(x,y)}{\partial(v_1,v_2)}$ at p. Therefore, by the Inverse Function Theorem, $(\pi \circ \mathsf{F}_\beta)^{-1}$ exists and is smooth near p. Since $\mathsf{F}_\beta^{-1} \circ \mathsf{F}_\alpha = (\pi \circ \mathsf{F}_\beta)^{-1} \circ (\pi \circ \mathsf{F}_\alpha)$, and all of $(\pi \circ \mathsf{F}_\beta)^{-1}$, π and F_α are smooth maps, their composition is also a smooth map. We have proved $\mathsf{F}_\beta^{-1} \circ \mathsf{F}_\alpha$ is smooth near p. Since p is arbitrary, $\mathsf{F}_\beta^{-1} \circ \mathsf{F}_\alpha$ is in fact smooth on the domain $\mathsf{F}_\alpha^{-1}(\mathcal{W})$. □

Exercise 1.14. Rewrite the proof of Proposition 1.11, *mutatis mutandis*, by assuming $\det \frac{\partial(y,z)}{\partial(u_1,u_2)}(p) \neq 0$ instead.

1.4. Maps and Functions from Surfaces

Let M be a regular surface in $\mathbb{R}^3$ with a smooth local parametrization $\mathsf{F}(u_1, u_2) : \mathcal{U} \to M$. Then, for any $p \in \mathsf{F}(\mathcal{U})$, one can define the partial derivatives for a function $f : M \to \mathbb{R}$ at p as follows. The subtle issue is that the domain of f is the surface M, but by pre-composing f with F, i.e. $f \circ \mathsf{F}$, one can regard it as a map from $\mathcal{U} \subset \mathbb{R}^2$ to $\mathbb{R}$. With a little abuse of notations, we denote:

$$\frac{\partial f}{\partial u_j}(p) := \frac{\partial (f \circ \mathsf{F})}{\partial u_j}(u_1, u_2)$$

where (u_1, u_2) is the point corresponding to p, i.e. $\mathsf{F}(u_1, u_2) = p$.

Remark 1.12. Note that $\frac{\partial f}{\partial u_j}(p)$ is defined locally on $\mathsf{F}(\mathcal{U})$, and depends on the choice of local parametrization F near p. □

Definition 1.13 (Functions of Class C^k)**.** Let M be a regular surface in $\mathbb{R}^3$, and $f : M \to \mathbb{R}$ be a function defined on M. We say f is C^k at $p \in M$ if for any smooth local parametrization $\mathsf{F} : \mathcal{U} \to M$ with $p \in \mathsf{F}(\mathcal{U})$, the composition $f \circ \mathsf{F}$ is C^k at (u_1, u_2) corresponding to p.

If f is C^k at p for any $p \in M$, then we say that f is a C^k function on M. Here k can be taken to be ∞, and in such case we call f to be a C^∞ (or smooth) function.

Remark 1.14. Although we require $f \circ \mathsf{F}$ to be C^k at $p \in M$ for *any* local parametrization F in order to say that f is C^k, by Proposition 1.11 it suffices to show that $f \circ \mathsf{F}$ is C^k at p for *at least one* F near p. It is because

$$f \circ \widetilde{\mathsf{F}} = (f \circ \mathsf{F}) \circ (\mathsf{F}^{-1} \circ \widetilde{\mathsf{F}})$$

and compositions of C^k maps (between Euclidean spaces) are C^k. □

Example 1.15. Let M be a regular surface in $\mathbb{R}^3$, then each of the x, y and z coordinates in $\mathbb{R}^3$ can be regarded as a function from M to $\mathbb{R}$. For any smooth local parametrization $\mathsf{F} : \mathcal{U} \to M$ around p given by

$$\mathsf{F}(u_1, u_2) = (x(u_1, u_2), y(u_1, u_2), z(u_1, u_2)),$$

we have $x \circ \mathsf{F}(u_1, u_2) = x(u_1, u_2)$. Since F is C^∞, we get $x \circ \mathsf{F}$ is C^∞ as well. Therefore, the coordinate functions x, y and z for any regular surface is smooth. □

Example 1.16. Let $f : M \to \mathbb{R}$ be the function from a regular surface M in $\mathbb{R}^3$ defined by:

$$f(p) := |p - p_0|^2$$

where $p_0 = (x_0, y_0, z_0)$ is a fixed point of $\mathbb{R}^3$. Suppose $\mathsf{F}(u, v)$ is a local parametrization of M. We want to compute $\frac{\partial f}{\partial u}$ and $\frac{\partial f}{\partial v}$.

Write $(x, y, z) = \mathsf{F}(u, v)$ so that x, y and z are functions of (u, v). Then

$$\begin{aligned}
\frac{\partial f}{\partial u} &:= \frac{\partial}{\partial u}(f \circ \mathsf{F}) \\
&= \frac{\partial}{\partial u} f(x(u,v), y(u,v), z(u,v)) \\
&= \frac{\partial}{\partial u}\Big((x(u,v) - x_0)^2 + (y(u,v) - y_0)^2 + (z(u,v) - z_0)^2\Big) \\
&= 2(x - x_0)\frac{\partial x}{\partial u} + 2(y - y_0)\frac{\partial y}{\partial u} + 2(z - z_0)\frac{\partial z}{\partial u}
\end{aligned}$$

Note that we can differentiate x, y and z by u because $\mathsf{F}(u, v)$ is smooth. Similarly, we have:

$$\frac{\partial f}{\partial v} = 2(x - x_0)\frac{\partial x}{\partial v} + 2(y - y_0)\frac{\partial y}{\partial v} + 2(z - z_0)\frac{\partial z}{\partial v}.$$

Again since $\mathsf{F}(u, v)$ (and hence x, y and z) is a smooth function of (u, v), we can differentiate $\frac{\partial f}{\partial u}$ and $\frac{\partial f}{\partial v}$ as many times as we wish. This concludes that f is a smooth funciton. $\square$

Exercise 1.15. Let $p_0(x_0, y_0, z_0)$ be a point in $\mathbb{R}^3$ and let $f(p) = |p - p_0|$ be the Euclidean distance between p and p_0 in $\mathbb{R}^3$. Suppose M is a regular surface in $\mathbb{R}^3$, one can then restrict the domain of f to M and consider it as a function:

$$f : M \to \mathbb{R}$$
$$p \mapsto |p - p_0|$$

Under what condition is the function $f : M \to \mathbb{R}$ smooth?

Now let M and N be two regular surfaces in $\mathbb{R}^3$. Then, one can also talk about mappings $\Phi : M \to N$ between them. In this section, we will define the notion of smooth maps between two surfaces.

Suppose $\mathsf{F} : \mathcal{U}_M \to M$ and $\mathsf{G} : \mathcal{U}_N \to N$ are two smooth local parametrizations of M and N respectively. One can then consider the composition $\mathsf{G}^{-1} \circ \Phi \circ \mathsf{F}$ after shrinking the domain. It is then a map between open subsets of $\mathbb{R}^2$.

However, in order for this composition to be well-defined, we require the image of $\Phi \circ \mathsf{F}$ to be contained in the image of G, which is not always guaranteed. Let $\mathcal{W} := \Phi(\mathcal{O}_M) \cap \mathcal{O}_N$ be the overlapping region on N of these two images. Then, provided that $\mathcal{W} \neq \emptyset$, the composition $\mathsf{G}^{-1} \circ \Phi \circ \mathsf{F}$ becomes well-defined as a map on:

$$\mathsf{G}^{-1} \circ \Phi \circ \mathsf{F} : (\Phi \circ \mathsf{F})^{-1}(\mathcal{W}) \to \mathcal{U}_N.$$

From now on, whenever we talk about this composition $\mathsf{G}^{-1} \circ \Phi \circ \mathsf{F}$, we always implicitly assume that $\mathcal{W} \neq \emptyset$ and its domain is $(\Phi \circ \mathsf{F})^{-1}(\mathcal{W})$.

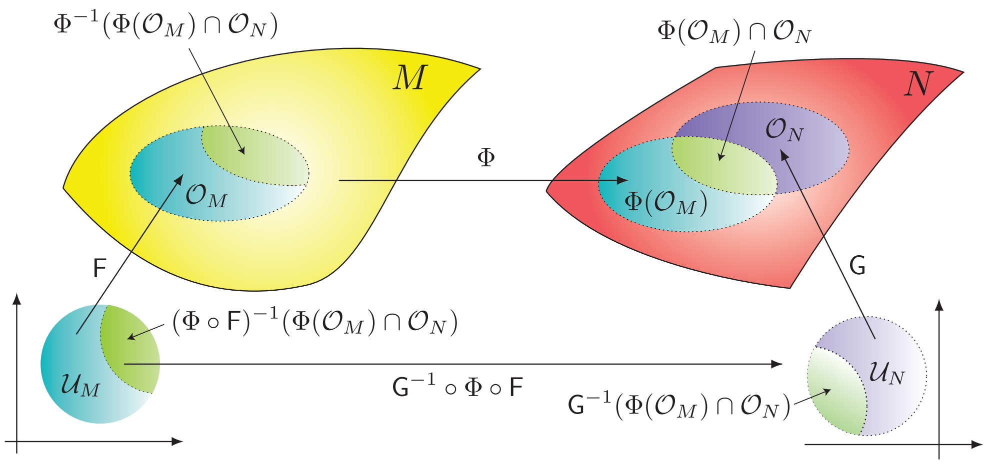


Figure 1.9. maps between regular surfaces

Definition 1.17 (Maps of Class C^k). Let M and N be two regular surfaces in $\mathbb{R}^3$, and $\Phi : M \to N$ be a map between them. We say Φ is C^k at $p \in M$ if for any smooth local parametrization $\mathsf{F} : \mathcal{U}_M \to M$ with $p \in \mathsf{F}(\mathcal{U}_M)$, and $\mathsf{G} : \mathcal{U}_N \to N$ with $\Phi(p) \in \mathsf{G}(\mathcal{U}_N)$, the composition $\mathsf{G}^{-1} \circ \Phi \circ \mathsf{F}$ is C^k at $\mathsf{F}^{-1}(p)$ as a map between subsets of $\mathbb{R}^2$.

If Φ is C^k at p for any $p \in M$, then we say that Φ is C^k on M. Here k can be taken to be ∞, and in such case we call Φ to be C^∞ (or smooth) on M.

Remark 1.18. Although we require $\mathsf{G}^{-1} \circ \Phi \circ \mathsf{F}$ to be C^k at $p \in M$ for *any* local parametrizations F and G in order to say that Φ is C^k, by Proposition 1.11 it suffices to show that $\mathsf{G}^{-1} \circ \Phi \circ \mathsf{F}$ is C^k at p for *at least one* pair of F and G. It is because

$$\widetilde{\mathsf{G}}^{-1} \circ \Phi \circ \widetilde{\mathsf{F}} = (\widetilde{\mathsf{G}}^{-1} \circ \mathsf{G}) \circ (\mathsf{G}^{-1} \circ \Phi \circ \mathsf{F}) \circ (\mathsf{F}^{-1} \circ \widetilde{\mathsf{F}})$$

and compositions of C^k maps (between Euclidean spaces) are C^k. □

Example 1.19. Let $\mathbb{S}^2$ be the unit sphere in $\mathbb{R}^3$. Consider the antipodal map $\Phi : \mathbb{S}^2 \to \mathbb{S}^2$ taking P to $-P$. In Example 1.4, two of the local parametrizations are given by:

$$\mathsf{F}_1(u_1, u_2) = \left(u_1,\ u_2,\ \sqrt{1 - u_1^2 - u_2^2}\right) : B_1(0) \subset \mathbb{R}^2 \to \mathbb{S}^2_+$$
$$\mathsf{F}_2(v_1, v_2) = \left(v_1,\ v_2,\ -\sqrt{1 - v_1^2 - v_2^2}\right) : B_1(0) \subset \mathbb{R}^2 \to \mathbb{S}^2_-$$

where $B_1(0)$ is the open unit disk in $\mathbb{R}^2$ centered at the origin, and $\mathbb{S}^2_+$ and $\mathbb{S}^2_-$ are the upper and lower hemispheres of $\mathbb{S}^2$ respectively. One can compute that:

$$\begin{aligned}\mathsf{F}_2^{-1} \circ \Phi \circ \mathsf{F}_1(u_1, u_2) &= \mathsf{F}_2^{-1} \circ \Phi \left(u_1,\ u_2,\ \sqrt{1 - u_1^2 - u_2^2}\right) \\ &= \mathsf{F}_2^{-2} \left(-u_1,\ -u_2,\ -\sqrt{1 - u_1^2 - u_2^2}\right) \\ &= (-u_1,\ -u_2)\end{aligned}$$

Clearly, the map $(u_1, u_2) \mapsto (-u_1, -u_2)$ is C^∞. It shows the antipodal map Φ is C^∞ at every point in $\mathsf{F}_1(B_1(0))$. One can show in similar way using other local parametrizations that Φ is C^∞ at points on $\mathbb{S}^2$ not covered by F_1.

Note that, for instance, the images of $\Phi \circ \mathsf{F}_1$ and F_1 are disjoint, and so $\mathsf{F}_1^{-1} \circ \Phi \circ \mathsf{F}_1$ is not well-defined. We don't need to verify whether it is smooth. □

Exercise 1.16. Let Φ be the antipodal map considered in Example 1.19, and F_+ and F_- be the two stereographic parametrizations of $\mathbb{S}^2$ defined in Example 1.5. Compute the maps $\mathsf{F}_+^{-1} \circ \Phi \circ \mathsf{F}_+$, $\mathsf{F}_-^{-1} \circ \Phi \circ \mathsf{F}_+$, $\mathsf{F}_+^{-1} \circ \Phi \circ \mathsf{F}_-$ and $\mathsf{F}_-^{-1} \circ \Phi \circ \mathsf{F}_-$. State their domains, and verify that they are smooth on their domains.

Exercise 1.17. Denote $\mathbb{S}^2$ to be the unit sphere $x^2 + y^2 + z^2 = 1$. Let $\Phi : \mathbb{S}^2 \to \mathbb{S}^2$ be the rotation map about the z-axis defined by:

$$\Phi(x, y, z) = (x \cos\alpha - y \sin\alpha, x \sin\alpha + y \cos\alpha, z)$$

where α is a fixed angle. Show that Φ is smooth.

Let M and N be two regular surfaces. If a map $\Phi : M \to N$ is C^∞, invertible, with C^∞ inverse map $\Phi^{-1} : N \to M$, then we say:

Definition 1.20 (Diffeomorphisms)**.** A map $\Phi : M \to N$ between two regular surfaces M and N in $\mathbb{R}^3$ is said to be a *diffeomorphism* if Φ is C^∞ and invertible, and also the inverse map Φ^{-1} is C^∞. If such a map Φ exists between M and N, then we say the surfaces M and N are *diffeomorphic*.

Example 1.21. The antipodal map $\Phi : \mathbb{S}^2 \to \mathbb{S}^2$ described in Example 1.19 is a diffeomorphism between $\mathbb{S}^2$ and itself. $\square$

Example 1.22. The sphere $x^2 + y^2 + z^2 = 1$ and the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ are diffeomorphic, under the map $\Phi(x, y, z) = (ax, by, cz)$ restricted on $\mathbb{S}^2$. $\square$

Exercise 1.18. Given any pair of C^∞ functions $f, g : \mathbb{R}^2 \to \mathbb{R}$, show that the graphs Γ_f and Γ_g are diffeomorphic.

Exercise 1.19. Show that $\Phi : \mathbb{S}^2 \to \mathbb{S}^2$ defined in Exercise 1.17 is a diffeomorphism.

1.5. Tangent Planes and Tangent Maps

1.5.1. Tangent Planes of Regular Surfaces. The tangent plane is an important geometric object associated to a regular surface. Condition (3) of a smooth local parametrization $\mathsf{F}(u,v)$ requires that the cross-product $\frac{\partial \mathsf{F}}{\partial u} \times \frac{\partial \mathsf{F}}{\partial v}$ is non-zero for any (u,v) in the domain, or equivalently, both tangent vectors $\frac{\partial \mathsf{F}}{\partial u}$ and $\frac{\partial \mathsf{F}}{\partial v}$ must be non-zero vectors and they are non-parallel to each other.

Therefore, the two vectors $\frac{\partial \mathsf{F}}{\partial u}$ and $\frac{\partial \mathsf{F}}{\partial v}$ span a two-dimensional subspace in $\mathbb{R}^3$. We call this subspace the *tangent plane*, which is defined rigorously as follows:

Definition 1.23 (Tangent Plane)**.** Let M be a regular surface in $\mathbb{R}^3$ and p be a point on M. Suppose $\mathsf{F}(u,v) : \mathcal{U} \subset \mathbb{R}^2 \to M$ is a smooth local parametrization around p, then the *tangent plane* at p, denoted by T_pM, is defined as follows:

$$T_pM := \operatorname{span}\left\{\frac{\partial \mathsf{F}}{\partial u}(p), \frac{\partial \mathsf{F}}{\partial v}(p)\right\} = \left\{a\frac{\partial \mathsf{F}}{\partial u}(p) + b\frac{\partial \mathsf{F}}{\partial v}(p) : a, b \in \mathbb{R}\right\}.$$

Here we have abused the notations for simplicity: $\frac{\partial \mathsf{F}}{\partial u}(p)$ means $\frac{\partial \mathsf{F}}{\partial u}$ evaluated at $(u,v) = \mathsf{F}^{-1}(p)$. Similarly for $\frac{\partial \mathsf{F}}{\partial v}(p)$.

Rigorously, T_pM is a plane passing through the *origin* while $p + T_pM$ is the plane tangent to the surface at p (see Figure 1.10). The difference between T_pM and $p + T_pM$ is very subtle, and we will almost neglect this difference.

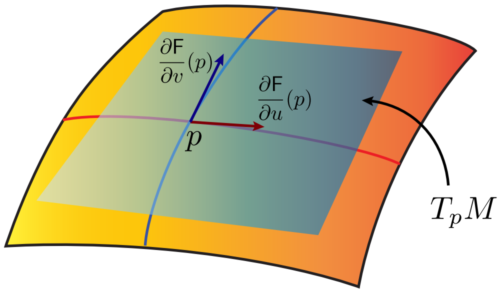


Figure 1.10. Tangent plane $p + T_pM$ at $p \in M$

Exercise 1.20. Show that the equation of the tangent plane $p + T_pM$ of the graph of a smooth function $f(x,y)$ at $p = (x_0, y_0, f(x_0,y_0))$ is given by:

$$z = f(x_0,y_0) + \left.\frac{\partial f}{\partial x}\right|_{(x_0,y_0)} (x - x_0) + \left.\frac{\partial f}{\partial y}\right|_{(x_0,y_0)} (y - y_0)$$

Exercise 1.21. [**dC76**, P.88] Consider the surface M given by $z = xf(y/x)$, where $x \neq 0$ and f is a smooth function. Show that the tangent planes $p + T_pM$ must pass through the origin $(0,0,0)$.

1.5.2. Tangent Maps between Regular Surfaces. Given a smooth map $\Phi : M \to N$ between two regular surfaces M and N, there is a naturally defined map called the *tangent map*, denoted by Φ_* in this course, between the tangent planes T_pM and $T_{\Phi(p)}N$.

Let us consider a smooth local parametrization $\mathsf{F}(u_1, u_2) : \mathcal{U}_M \to M$. The composition $\Phi \circ \mathsf{F}$ can be regarded as a map from $\mathcal{U}_M$ to $\mathbb{R}^3$, so one can talk about its partial derivatives $\frac{\partial(\Phi \circ \mathsf{F})}{\partial u_i}$:

$$\frac{\partial \Phi}{\partial u_i}(\Phi(p)) := \left.\frac{\partial(\Phi \circ \mathsf{F})}{\partial u_i}\right|_{(u_1,u_2)} = \left.\frac{d}{dt}\right|_{t=0} \Phi \circ \mathsf{F}((u_1, u_2) + t\mathsf{e}_i)$$

where (u_1, u_2) is a point in $\mathcal{U}_M$ such that $\mathsf{F}(u_1, u_2) = p$. The curve $\mathsf{F}((u_1, u_2) + t\mathsf{e}_i)$ is a curve on M with parameter t along the u_i-direction. The curve $\Phi \circ \mathsf{F}((u_1, u_2) + t\mathsf{e}_i)$ is then the image of the u_i-curve of M under the map Φ (see Figure 1.11). It is a curve on N so $\frac{\partial \Phi}{\partial u_i}$ which is a tangent vector to the surface N.

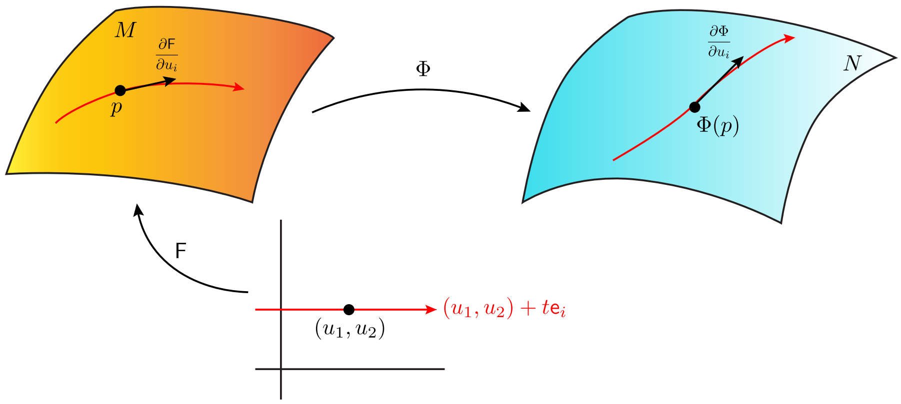


Figure 1.11. Partial derivative of the map $\Phi : M \to N$

Exercise 1.22. Denote $\mathbb{S}^2$ to be the unit sphere $x^2 + y^2 + z^2 = 1$. Let $\Phi : \mathbb{S}^2 \to \mathbb{S}^2$ be the rotation map about the z-axis defined by:

$$\Phi(x, y, z) = (x \cos \alpha - y \sin \alpha, x \sin \alpha + y \cos \alpha, z)$$

where α is a fixed angle. Calculate the following partial derivatives under the given local parametrizations:

(a) $\frac{\partial \Phi}{\partial \theta}$ and $\frac{\partial \Phi}{\partial \varphi}$ under $\mathsf{F}(\theta, \varphi) = (\sin \varphi \cos \theta, \sin \varphi \sin \theta, \cos \varphi)$;

(b) $\frac{\partial \Phi}{\partial u}$ and $\frac{\partial \Phi}{\partial v}$ under F_2 in Example 1.4;

(c) $\frac{\partial \Phi}{\partial u}$ and $\frac{\partial \Phi}{\partial v}$ under F_+ in Example 1.5.

Next, we write the partial derivative $\frac{\partial \Phi}{\partial u_i}$ in a *fancy* way. Define:

$$\Phi_*\left(\frac{\partial \mathsf{F}}{\partial u_i}\right) := \frac{\partial \Phi}{\partial u_i}.$$

Then, one can regard Φ_* as a map that takes the tangent vector $\frac{\partial \mathsf{F}}{\partial u_i}$ in T_pM to another vector $\frac{\partial \Phi}{\partial u_j}$ in $T_{\Phi(p)}N$. Since $\left\{\frac{\partial \mathsf{F}}{\partial u_i}(p)\right\}$ is a basis of T_pM, one can then extend Φ_* linearly and define it as the *tangent map* of Φ. Precisely, we have:

Definition 1.24 (Tangent Maps)**.** Let $\Phi : M \to N$ be a smooth map between two regular surfaces M and N in $\mathbb{R}^3$. Let $\mathsf{F} : \mathcal{U}_M \to M$ and $\mathsf{G} : \mathcal{U}_N \to N$ be two smooth local parametrizations covering p and $\Phi(p)$ respectively. Then, the *tangent map* of Φ at $p \in M$ is denoted by $(\Phi_*)_p$ and is defined as:

$$(\Phi_*)_p : T_pM \to T_{\Phi(p)}N$$
$$(\Phi_*)_p \left(\sum_{i=1}^{2} a_i \frac{\partial \mathsf{F}}{\partial u_i}(p) \right) = \sum_{i=1}^{2} a_i \frac{\partial \Phi}{\partial u_i}(\Phi(p))$$

If the point p is clear from the context, $(\Phi_*)_p$ can be simply denoted by Φ_*.

Remark 1.25. Some textbooks may use $d\Phi_p$ to denote the tangent map of Φ at p. $\square$

Example 1.26. Consider the unit sphere $\mathbb{S}^2$ locally parametrized by

$$\mathsf{F}(\theta, \varphi) = (\sin\varphi\cos\theta, \sin\varphi\sin\theta, \cos\varphi)$$

and the rotation map:

$$\Phi(x, y, z) = (x\cos\alpha - y\sin\alpha, x\sin\alpha + y\cos\alpha, z)$$

From Exercise 1.22, one should have figured out that:

$$\frac{\partial \Phi}{\partial \theta} = (-\sin\varphi\sin(\theta+\alpha), \sin\varphi\cos(\theta+\alpha), 0)$$
$$\frac{\partial \Phi}{\partial \varphi} = (\cos\varphi\cos(\theta+\alpha), \cos\varphi\sin(\theta+\alpha), -\sin\varphi)$$

Next we want to write them in terms of the basis $\left\{ \frac{\partial \mathsf{F}}{\partial \theta}, \frac{\partial \mathsf{F}}{\partial \varphi} \right\}$. However, we should be careful about the base points of these vectors. Consider a point $p \in \mathbb{S}^2$ with local coordinates (θ, φ), the vectors $\frac{\partial \Phi}{\partial \theta}$ and $\frac{\partial \Phi}{\partial \varphi}$ computed above are based at the point $\Phi(p)$ with local coordinates $(\theta + \alpha, \varphi)$. Therefore, we should express them in terms of the basis $\left\{ \frac{\partial \mathsf{F}}{\partial \theta}(\Phi(p)), \frac{\partial \mathsf{F}}{\partial \varphi}(\Phi(p)) \right\}$, not $\left\{ \frac{\partial \mathsf{F}}{\partial \theta}(p), \frac{\partial \mathsf{F}}{\partial \varphi}(p) \right\}$!

At $\Phi(p)$, we have:

$$\frac{\partial \mathsf{F}}{\partial \theta}(\Phi(p)) = (-\sin\varphi\sin(\theta+\alpha), \sin\varphi\cos(\theta+\alpha), 0) = \frac{\partial \Phi}{\partial \theta}(\Phi(p))$$
$$\frac{\partial \mathsf{F}}{\partial \varphi}(\Phi(p)) = (\cos\varphi\cos(\theta+\alpha), \cos\varphi\sin(\theta+\alpha), -\sin\varphi) = \frac{\partial \Phi}{\partial \varphi}(\Phi(p))$$

Therefore, the tangent map $(\Phi_*)_p$ acts on the basis vectors by:

$$(\Phi_*)_p \left(\frac{\partial \mathsf{F}}{\partial \theta}(p) \right) = \frac{\partial \mathsf{F}}{\partial \theta}(\Phi(p))$$
$$(\Phi_*)_p \left(\frac{\partial \mathsf{F}}{\partial \varphi}(p) \right) = \frac{\partial \mathsf{F}}{\partial \varphi}(\Phi(p))$$

In other words, the matrix representation $[(\Phi_*)_p]$ with respect to the bases

$$\left\{ \frac{\partial \mathsf{F}}{\partial \theta}(p), \frac{\partial \mathsf{F}}{\partial \varphi}(p) \right\} \text{ for } T_p\mathbb{S}^2 \qquad \left\{ \frac{\partial \mathsf{F}}{\partial \theta}(\Phi(p)), \frac{\partial \mathsf{F}}{\partial \varphi}(\Phi(p)) \right\} \text{ for } T_{\Phi(p)}\mathbb{S}^2$$

is the identity matrix. However, it is not perfectly correct to say $(\Phi_*)_p$ is an identity map, since the domain and co-domain are different tangent planes. $\square$

Exercise 1.23. Let Φ be as in Example 1.26. Consider the stereographic parametrization $\mathsf{F}_+(u,v)$ defined in Example 1.5. Suppose $p \in \mathbb{S}^2$, express the matrix representation $[(\Phi_*)_p]$ with respect to the bases $\left\{\frac{\partial \mathsf{F}_+}{\partial u}, \frac{\partial \mathsf{F}_+}{\partial v}\right\}_p$ and $\left\{\frac{\partial \mathsf{F}_+}{\partial u}, \frac{\partial \mathsf{F}_+}{\partial v}\right\}_{\Phi(p)}$

1.5.3. Tangent Maps and Jacobian Matrices. Let $\Phi : M \to N$ be a smooth map between two regular surfaces. Instead of computing the matrix representation of the tangent map Φ_* directly by taking partial derivatives (c.f. Example 1.26), one can also find it out by computing a Jacobian matrix.

Suppose $\mathsf{F}(u_1,u_2) : \mathcal{U}_M \to M$ and $\mathsf{G}(v_1,v_2) : \mathcal{U}_N \to N$ are local parametrizations of M and N. The composition $\mathsf{G}^{-1} \circ \Phi \circ \mathsf{F}$ can be regarded as a map between the u_1u_2-plane to the v_1v_2-plane. As such, one can write

$$\mathsf{G}^{-1} \circ \Phi \circ \mathsf{F}(u_1,u_2) = (v_1(u_1,u_2), v_2(u_1,u_2)).$$

By considering $\Phi \circ \mathsf{F}(u_1,u_2) = \mathsf{G}(v_1(u_1,u_2), v_2(u_1,u_2))$, one can differentiate both sides with respect to u_i:

$$\frac{\partial}{\partial u_i}(\Phi \circ \mathsf{F}) = \frac{\partial}{\partial u_i}\mathsf{G}(v_1(u_1,u_2), v_2(u_1,u_2)) = \sum_{k=1}^{2} \frac{\partial \mathsf{G}}{\partial v_k}\frac{\partial v_k}{\partial u_i}. \tag{1.1}$$

Here we used the chain rule. Note that $\left\{\frac{\partial \mathsf{G}}{\partial v_k}\right\}$ is a basis for $T_{\Phi(p)}N$.

Using (1.1), one can see:

$$\Phi_*\left(\frac{\partial \mathsf{F}}{\partial u_1}\right) := \frac{\partial \Phi}{\partial u_1} = \frac{\partial}{\partial u_1}(\Phi \circ \mathsf{F}) = \frac{\partial v_1}{\partial u_1}\frac{\partial \mathsf{G}}{\partial v_1} + \frac{\partial v_2}{\partial u_1}\frac{\partial \mathsf{G}}{\partial v_2}$$
$$\Phi_*\left(\frac{\partial \mathsf{F}}{\partial u_2}\right) := \frac{\partial \Phi}{\partial u_2} = \frac{\partial}{\partial u_2}(\Phi \circ \mathsf{F}) = \frac{\partial v_1}{\partial u_2}\frac{\partial \mathsf{G}}{\partial v_1} + \frac{\partial v_2}{\partial u_2}\frac{\partial \mathsf{G}}{\partial v_2}$$

Hence the matrix representation of $(\Phi_*)_p$ with respect to the bases $\left\{\frac{\partial \mathsf{F}}{\partial u_i}(p)\right\}$ and $\left\{\frac{\partial \mathsf{G}}{\partial v_i}(\Phi(p))\right\}$ is the Jacobian matrix:

$$\left.\frac{\partial(v_1,v_2)}{\partial(u_1,u_2)}\right|_{\mathsf{F}^{-1}(p)} = \begin{bmatrix} \frac{\partial v_1}{\partial u_1} & \frac{\partial v_1}{\partial u_2} \\ \frac{\partial v_2}{\partial u_1} & \frac{\partial v_2}{\partial u_2} \end{bmatrix}_{\mathsf{F}^{-1}(p)}$$

Example 1.27. Let $\Phi : \mathbb{S}^2 \to \mathbb{S}^2$ be the rotation map as in Example 1.26. Consider again the local parametrization:

$$\mathsf{F}(\theta,\varphi) = (\sin\varphi\cos\theta, \sin\varphi\sin\theta, \cos\varphi).$$

By standard trigonometry, one can find out that $\Phi(\mathsf{F}(\theta,\varphi)) = \mathsf{F}(\theta+\alpha,\varphi)$. Equivalently, the map $\mathsf{F}^{-1} \circ \Phi \circ \mathsf{F}$ (in a suitable domain) is the map:

$$(\theta,\varphi) \mapsto (\theta+\alpha,\varphi).$$

As α is a constant, the Jacobian matrix of $\mathsf{F}^{-1} \circ \Phi \circ \mathsf{F}$ is the identity matrix, and so the matrix $[(\Phi_*)_p]$ with respect to the bases $\left\{\frac{\partial \mathsf{F}}{\partial \theta}, \frac{\partial \mathsf{F}}{\partial \varphi}\right\}_p$ and $\left\{\frac{\partial \mathsf{F}}{\partial \theta}, \frac{\partial \mathsf{F}}{\partial \varphi}\right\}_{\Phi(p)}$ is the identity matrix (which was also obtained by somewhat tedious computations in Example 1.26). □

Exercise 1.24. Do Exercise 1.23 by considering Jacobian matrices.