Power Comparison for T-Test and Sign Test- Normal and Exponential Samples

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What is the Sign Test

- Nonparametric alternative to t-test when normality is not assumed (Cannot use mean for hypothesis test as distribution not symmetric)

  i.e., $X_1, X_2 .... X_n \sim Unknown\ Distribution$

- Sign test is used for hypothesis tests for a median ($\mu = median$)

$$H_0\colon\ \mu = \mu_0$$
$$H_1\colon \mu > \mu_0$$

- For our test

$$H_0\colon\ \mu = 0$$
$$H_1\colon \mu > 0$$

- Create a new variable Y

$$X_i = 0\ then\ Y_i = 1$$
$$X_i > 0\ then\ Y_i = 0$$

- Where $Y_i \sim Bern(\frac{1}{2})$, where $p = \frac{1}{2}$ under the null hypothesis

How to Calculate the Sign Test Statistic W

- $W = \sum_{i=0}^{n} Y_i$
- $W \leq c \; reject \; H_0 \; and \; accept \; when \; W > c$
- c is the critical value of x where the CDF of the Binomial Distribution for $Y_i \sim Bin\left(n, \frac{1}{2}\right)$ has a probability less than or equal to α
- $\sum_{w=0}^{c} \binom{n}{w} (\frac{1}{2})^n \leq \alpha_0 < \sum_{w=0}^{c+1} \binom{n}{w} (\frac{1}{2})^n$
- $F(c) \leq \alpha_0 < F(c+1)$, where F is the CDF of Binomial distribution

Example of Sign Test

$n = 50$ oberservations where $X_1, X_2 \ldots X_n \sim Unknown\ Distribution$

i	X_i	Y_i
1	0	1
2	3	0
3	4	0
4	2	0
5	1	0
…..		
50	0	1

$H_0 : \mu = 0$

$H_a : \mu > 0$

$\alpha = .05$

μ *is the median*

$$\sum_{w=0}^{c} \binom{n}{w} \left(\frac{1}{2}\right)^n \leq \alpha_0 < \sum_{w=0}^{c+1} \binom{n}{w} \left(\frac{1}{2}\right)^n$$

$W = 13$ *and* $c = 18$ (Calculated using R)

$W < c$ therefore reject the null hypothesis

Calculating Power for T Test

Power = Probability of rejecting the null hypothesis while the alternative is true

$$\pi(\theta \in \ \Omega_1 | \ \delta = 1) = Power$$

1. We need to calculate the rejection region for the underlying distribution

$Z_{1-\propto} = \frac{\hat{\mu}-\mu}{\frac{\sigma}{\sqrt{n}}}$ so that $\hat{\mu} = Z_{1-\alpha} * \frac{\sigma}{\sqrt{n}} + \mu$

2. For the null H_0: $\mu = 0$, the t distribution is as follows

$$t = \frac{\hat{\mu}-0}{\frac{s}{\sqrt{n}}}$$

3. Calculate the probability of rejecting null hypothesis for t

$$P\left(t > t_{n-1,1-\propto} \middle| \ \mu = \mu_0 > 0\right)$$

Calculating Power for Sign Test

Power = Probability of rejecting the null hypothesis while the alternative is true

$$Power = p(w \leq c)$$

1. Find the probability that X_i is less than or equal to μ_0 for its respective underlying distribution. For $H_0: \mu = \mu_0$

$$p = \mathrm{P}(X_i \leq 0)$$

2. Plug in p into the Sign Power Function to calculate Power

$$Power = p(w \leq c) = \sum_{w=0}^{c} \binom{n}{w} p^w (1-p)^{n-w}$$

Power Comparisons for $X_1, X_2 \ldots X_n \sim N(\mu, \sigma^2)$

- Goal: To find the test with the highest power for various values of μ and n using R simulation
- $\mu < 0$ values were not used as power would be zero
- Conclusion- T test has higher power for all n

$\mu = .5$

n	$Power\ t$	$Power\ Sign$
10	.1123249	.02932231
20	.1951637	.07339428
30	.2304711	.17757490
40	.2710237	.18181277
50	.3793789	.18164215

$\mu = 1$

n	$Power\ t$	$Power\ Sign$
10	.2569638	.06815736
20	.3334795	.19240541
30	.4241432	.41916884
40	.6417767	.46953730
50	.7941511	.50839380

$\mu = 3$

n	$Power\ t$	$Power\ Sign$
10	.9570575	.5128569
20	.9984613	.9158307
30	.9989558	.9954012
40	.9996289	.9992399
50	.9999996	.9998710

Power Comparisons for $X_1, X_2 \ldots X_n \sim Exp(\mu)$

- Goal: To find the test with the highest power for various values of μ and n using R simulation
- Sign Test has the higher power of 1 for all n and μ as $p = \mathrm{P}(X_i \leq$

$\mu = .5$

n	$Power\ t$	$Power\ Sign$
10	.5437356	1
20	.9774222	1
30	.8706834	1
40	.9891579	1
50	.9992951	1

$\mu = 1$

n	$Power\ t$	$Power\ Sign$
10	.9502959	1
20	.9993981	1
30	.9986037	1
40	.9954691	1
50	.9985689	1