CS 131 – Spring 2017 – Lab 4

Question 1 Consider the following incorrect theorem:
Incorrect Theorem. *Suppose n is a natural number larger than 2, and n is not a prime number. Then 2n+13 is not a prime number.*
What are the hypotheses and conclusion of this theorem? Show that the theorem is incorrect by finding a counterexample.
Solution *The hypotheses are n is a natural number, n is larger than 2, and n is not prime. The conclusion is $2n + 13$ is not prime. Let's make a table to find a counterexample:*

n	$n > 2$	n not prime	$2n+13$	$2n+13$ not prime
0	F	T	13	F
1	F	T	15	T
2	F	F	17	F
3	T	F	19	F
4	T	T	21	T
5	T	F	23	F
6	T	T	25	T
7	T	F	27	T
8	T	T	29	F

As we can see from the table, $n = 8$ is a counterexample. 8 is a natural number larger than 2 which is not prime, but for which $2n + 13 = 29$ is prime. So, for $n = 8$ all of the hypotheses of this statement are satisfied but the conclusion does not hold.

Question 2 Suppose a and b are real numbers. Prove that if $a < b < 0$ then $a^2 > b^2$.
Solution
Scratch Work
First state the givens and goal.

Givens	Goal
$a \in \mathbb{R}$	$(a < b < 0) \rightarrow a^2 > b^2$
$b \in \mathbb{R}$	

To prove $P \rightarrow Q$ we can assume P is true and prove Q. So, instead we can use the following givens and goal:

Givens	Goal
$a \in \mathbb{R}$	$a^2 > b^2$
$b \in \mathbb{R}$	
$(a < b < 0)$	

$a < b$ and both a and b are negative, so multiplying by a on both sides of the inequality gives $a^2 > ab$ and by b on both sides gives $ab > b^2$. We get $a^2 > ab > b^2$ from these two inequalities, and thus $a^2 > b^2$.

Proof
Suppose $a < b < 0$. Multiplying the inequality $a < b$ by the negative number a we can conclude that $a^2 > ab$, and similiarly multiplying by b we get $ab > b^2$. Therefore, $a^2 > ab > b^2$, so $a^2 > b^2$, as required. Thus, if $a < b < 0$ then $a^2 > b^2$.

Question 3 Suppose a, b, c, and d are real numbers, $0 < a < b$, and $d > 0$. Prove that if $ac \geq bd$ then $c > d$.

Solution

<u>*Scratch Work*</u>

First state the givens and goal.

Givens	Goal
$a \in \mathbb{R}$	$(ac \geq bd) \to (c > d)$
$b \in \mathbb{R}$	
$c \in \mathbb{R}$	
$d \in \mathbb{R}$	
$0 < a < b$	
$d > 0$	

To prove $P \to Q$ we can instead prove the contrapositive $\neg Q \to \neg P$. In order to prove $\neg Q \to \neg P$ we can assume $\neg Q$ and prove $\neg P$. So, instead we can use the following givens and goal:

Givens	Goal
$a \in \mathbb{R}$	$ac < bd$
$b \in \mathbb{R}$	
$c \in \mathbb{R}$	
$d \in \mathbb{R}$	
$0 < a < b$	
$d > 0$	
$c \leq d$	

Multiplying $c \leq d$ by positive number a gives $ac \leq ad$ and multiplying $a < b$ by positive number d gives $ad < bd$. Combining these gives $ac \leq ad < bd$ and so $ac < bd$.

<u>*Proof*</u>

We will prove the contrapositive. Suppose $c \leq d$. Multiplying both sides of this inequality by the positive number a, we get $ac \leq ad$. Also, multiplying both sides of the given equality $a < b$ by the positive number d gives us $ad < bd$. Combining $ac \leq ad$ and $ad < bd$, we can conclude that $ac < bd$. Thus, if $ac \geq bd$ then $c > d$.

Question 4 Suppose that $y + x = 2y - x$, and x and y are not both zero. Prove that $y \neq 0$.

Solution

<u>*Scratch Work*</u>

First state the givens and goal.

Givens	Goal
$y + x = 2y - x$	$y \neq 0$
$\neg(x = 0 \wedge y = 0)$	

A proof by contradiction seems appropriate here, since the goal is negated and we cannot easily express it in a positive form. So, we can use the following givens and goal instead:

Givens	Goal
$y + x = 2y - x$	Contradiction
$\neg(x = 0 \wedge y = 0)$	
$y = 0$	

Using the first and third given we get that $0 + x = 2(0) - x$ or $x = -x$, which implies that $x = 0$. This contradicts the second given though, since this means both x and y are 0. Thus, $y \neq 0$.

<u>*Proof*</u>

Suppose $y = 0$. Then from $y + x = 2y - x$ we get that $x = -x$. This implies that $x = 0$ as well, but this contradicts the fact that not both x and y are 0. Thus, $y \neq 0$.

Question 5 Use a direct proof to show that every odd integer is the difference of two squares.
Solution
<u>Scratch Work</u>
We can restate the statement as "if n is an odd integer, then n is the difference of two squares." Now we can state the givens and goal.

Givens	*Goal*
	n is odd integer $\rightarrow$ n is difference of two squares

Since the statement is an implication, we can assume the hypothesis and show the conclusion. So, the new givens and goal are:

Givens	*Goal*
n is odd integer	*n is difference of two squares*

Since n is odd, it can be written as $2k + 1$ for some integer k. Consider $(k + 1)^2 = k^2 + 2k + 1$. So, $n = 2k + 1 = (k + 1)^2 - k^2$ and thus, n is the difference of two squares, $k + 1$ and k.

<u>Proof</u> Let n be an odd integer. n can be written as $2k + 1$ for some integer k by the definition of odd. $2k + 1 = (k + 1)^2 - k^2$ and so, n is the difference of two squares ($k + 1$ and k). Thus, every odd integer is the difference of two squares.