CS 131 – Spring 2017 – Lab 5

Question 1 Prove that the sum of an irrational number and a rational number is irrational.
Solution *Let i be an irrational number and q be a rational number. Suppose their sum, $i + q$, is rational. Then $i + q = s$ where s is rational. So, $i = s - q = s + (-q)$. If q is a rational number then $-q$ is as well, and the sum of two rational numbers is still rational. Therefore, i is rational. But this contradicts our assumption that i is irrational. Thus, the sum of i and q must be irrational and the statement is proven.*

Question 2 Show that these statements about the integer x are equivalent: (i) $3x + 2$ is even, (ii) $x + 5$ is odd, (iii) x^2 is even.
Solution
We prove that all these are equivalent to x being even. If x is even, then $x = 2k$ for some integer k. Therefore $3x + 2 = 3(2k) + 2 = 6k + 2 = 2(3k + 1)$, which is even, because it has been written in the form $2t$, where $t = 3k + 1$. Similarly, $x + 5 = 2k + 5 = 2k + 4 + 1 = 2(k + 2) + 1$, so $x + 5$ is odd; and $x^2 = (2k)^2 = 2(2k^2)$, so x^2 is even. For the converses, we will use a proof by contraposition. So assume that x is not even; thus x is odd and we can write $x = 2k + 1$ for some integer k. Then $3x + 2 = 3(2k + 1) + 2 = 6k + 5 = 2(3k + 2) + 1$, which is odd (i.e., not even), because it has been written in the form $2t + 1$, where $t = 3k + 2$. Similarly, $x + 5 = 2k + 1 + 5 = 2(k + 3)$, so $x + 5$ is even (i.e., not odd). Lastly, $x^2 = (2k + 1)^2 = 4k^2 + 4k + 1 = 2(2k^2 + 2k) + 1$, so x^2 is odd.

Question 3 Prove the triangle inequality, which states that if x and y are real numbers, then $|x| + |y| \geq |x + y|$.
Solution
To prove this we will use a proof by cases. There are four cases.
Case 1: $x \geq 0$ and $y \geq 0$. Then $|x| + |y| = x + y = |x + y|$.
Case 2: $x < 0$ and $y < 0$. Then $|x| + |y| = -x + (-y) = -(x + y) = |x + y|$ because $x + y < 0$.
Case 3: $x \geq 0$ and $y < 0$. Then $|x| + |y| = x + (-y)$. If $x \geq -y$, then $|x + y| = x + y$. But because $y < 0$, $-y > y$, so $|x| + |y| = x + (-y) > x + y = |x + y|$. If $x < -y$, then $|x + y| = -(x + y) = -x + (-y)$. But because $x \geq 0$, $x \geq -x$, so $|x| + |y| = x + (-y) \geq -x + (-y) = |x + y|$.
Case 4: $x < 0$ and $y \geq 0$. Identical to Case 3 with the roles of x and y reversed.

Question 4 Prove that there are 100 consecutive positive integers that are not perfect squares. Is your proof constructive or nonconstructive?
Solution
10,001, 10,002, ..., 10,100 are all nonsquares, because $100^2 = 10,000$ and $101^2 =$ 10,201; constructive.