CS 131 – Spring 2017 – Lab 6

Question 1 Theorem. All horses are the same color.

Proof. We'll induct on the number of horses. Base case: 1 horse. Clearly with just 1 horse, all horses have the same color.
Now, for the inductive step: we'll show that if it is true for any group of N horses, that all have the same color, then it is true for any group of N+1 horses. Well, given any set of N+1 horses, if you exclude the last horse, you get a set of N horses. By the inductive step these N horses all have the same color. But by excluding the first horse in the pack of N+1 horses, you can conclude that the last N horses also have the same color. Therefore all N+1 horses have the same color. What's wrong with this proof by induction? Multiple answers are allowed.

- In the base step
- In the inductive step
- In the choice of base step
- In the choice of inductive hypothesis
- This proof is correct

Solution *Consider what happens in the inductive step when N+1=2. When N+1=2 we have a set of just 2 horses, say $\{h_1, h_2\}$. Excluding the last horse gives just $\{h_1\}$ and by the inductive hypothesis the statement holds for this set. Excluding the first horse gives just $\{h_2\}$ and again the statement holds for this set. However, there is no reason that combining the sets into $\{h_1, h_2\}$ must also have the same color since the first and second sets share no elements. Thus, the problem is in the inductive step. If it were possible to prove that any group of two horses has the same color we could add this to the basis to fix the proof, however this is not always true.*

Question 2 Let P(n) be the statement that $1^2 + 2^2 + \quad + n^2 = \frac{n(n+1)(2n+1)}{6}$ for the positive integer n.

a) What is the statement P(1)?

b) Show that P(1) is true, completing the basis step of the proof.

c) What is the inductive hypothesis?

d) What do you need to prove in the inductive step?

e) Complete the inductive step, identifying where you use the inductive hypothesis.

f) Explain why these steps show that this formula is true whenever n is a positive integer.

Solution

a) P(1) is the statement that $1^2 = \frac{1(1+1)(2\cdot 1+1)}{6}$.

b) $1^2 = 1 = \frac{6}{6} = \frac{1(1+1)(2\cdot 1+1)}{6}$

c) The inductive hypothesis is P(k) which is that $1^2 + 2^2 ... + k^2 = \frac{k(k+1)(2k+1)}{6}$.

d) You need to prove that P(k) implies P(k+1), where P(k) is the inductive hypothesis in part c) and P(k+1) is that $1^2 + 2^2 ... + k^2 + (k+1)^2 = \frac{(k+1)(k+2)(2(k+1)+1)}{6}$.

e) Assume the inductive hypothesis.

$$\begin{aligned}
1^2 + 2^2 ... + k^2 + (k+1)^2 &= \frac{k(k+1)(2k+1)}{6} + (k+1)^2 \text{ (by inductive hypothesis)} \\
&= (k+1)(\frac{k(2k+1)}{6} + (k+1)) \\
&= (k+1)(\frac{2k^2+k}{6} + \frac{6k+6}{6}) \\
&= (k+1)(\frac{2k^2+k+6k+6}{6}) \\
&= (k+1)(\frac{2k^2+7k+6}{6}) \\
&= (k+1)(\frac{(2k+3)(k+2)}{6}) \\
&= \frac{(k+1)(k+2)(2(k+1)+1)}{6}
\end{aligned}$$

f) We have completed both the basis and inductive step, so by the principle of mathematical induction the statement holds for all positive integers.

Question 3 Prove that $3 + 3 \cdot 5 + 3 \cdot 5^2 + ... + 3 \cdot 5^n = \frac{3(5^{n+1}-1)}{4}$ whenever n is a nonnegative integer.
Solution
Let P(n) be the statement that $3 + 3 \cdot 5 + 3 \cdot 5^2 + ... + 3 \cdot 5^n = \frac{3(5^{n+1}-1)}{4}$.
Basis: The basis is the true statement P(0): $3 = \frac{3(5^1-1)}{4}$.
Inductive Step: Assume the inductive hypothesis, P(k) for some nonnegative integer k, which is $3 + 3 \cdot 5 + 3 \cdot 5^2 + ... + 3 \cdot 5^k = \frac{3(5^{k+1}-1)}{4}$.

$$\begin{aligned}
3 + 3 \cdot 5 + 3 \cdot 5^2 + ... + 3 \cdot 5^k + 3 \cdot 5^{k+1} &= \frac{3(5^{k+1}-1)}{4} + 3 \cdot 5^{k+1} \\
&= \frac{3 \cdot 5^{k+1} - 3}{4} + \frac{12 \cdot 5^{k+1}}{4} \\
&= \frac{3 \cdot 5^{k+1} - 3 + 12 \cdot 5^{k+1}}{4} \\
&= \frac{15 \cdot 5^{k+1} - 3}{4} \\
&= \frac{3 \cdot 5^{k+2} - 3}{4} \\
&= \frac{3(5^{k+2}-1)}{4}
\end{aligned}$$

We have completed both the basis and inductive step, so by the principle of mathematical induction the statement holds for all nonnegative integers.

Question 4 Prove that $n^2 - 1$ is divisible by 8 whenever n is an odd positive integer.
Solution
Let P(n) be the statement that 8 divides $n^2 - 1$.
Basis: The basis case is the true statement P(1): 8 divides $1^2 - 1 = 0$.
Inductive Step: Assume $P(k)$ *for some odd positive integer* k. *Show* $P(k+2)$ *since the next odd positive integer after* k *is* $k+2$.

$$\begin{aligned}(k+2)^2 - 1 &= k^2 + 4k + 4 - 1 \\ &= (k^2 - 1) + 4(k+1)\end{aligned}$$

$k^2 - 1$ *is divisible by 8 by the inductive hypothesis.* $k+1$ *is even since* k *is odd, so* $4(k+1)$ *is also divisible by 8. Thus,* $(k+2)^2 - 1$ *is divisible by 8, as required. We have completed both the basis step and inductive step, so by the principle of mathematical induction the statement holds for all odd positive integers.*