CS 131 – Spring 2017 – Lab 7

Question 1 A vending machine dispensing books of stamps accepts only one-dollar coins, \$1 bills, and \$5 bills.

a) Find a recurrence relation for the number of ways to deposit n dollars in the vending machine, where the order in which the coins and bills are deposited matters.

b) What are the initial conditions?

c) How many ways are there to deposit \$10 for a book of stamps?

Solution

a) Let a_n be the number of ways to deposit n dollars in the vending machine. We have three choices for what to deposit first: a one-dollar coin, a \$1 bill or a \$5 bill. After we deposit a one-dollar coin or a \$1 bill there are a_{n-1} ways to deposit the rest of the money, and if we deposit a \$5 bill there are a_{n-5} ways to deposit the rest. So, $a_n = 2a_{n-1} + a_{n-5}$.

b) There is one way to deposit \$0, by inserting nothing. There are 2 ways to deposit \$1, insert a one-dollar coin or a \$1 bill. There are 4 ways to deposit \$2, for each of the dollars deposited there is a choice between a one-dollar coin or a \$1 bill. Similarly, there are 8 ways to deposit \$3 and 16 ways to deposit \$4. So, $a_0 = 1, a_1 = 2, a_2 = 4, a_3 = 8$ and $a_4 = 16$.

c) $a_5 = 2a_4 + a_0 = 2(16) + 1 = 33$
$a_6 = 2a_5 + a_1 = 2(33) + 2 = 68$
$a_7 = 2a_6 + a_2 = 2(68) + 4 = 140$
$a_8 = 2a_7 + a_3 = 2(140) + 8 = 288$
$a_9 = 2a_8 + a_4 = 2(288) + 16 = 592$
$a_10 = 2a_9 + a_5 = 2(592) + 33 = 1217$
So, there are 1217 ways to deposit \$10 into the vending machine.

Question 2

a) Find a recurrence relation for the number of bit strings of length n that contain two consecutive 0's.

b) What are the initial conditions?

c) How many bit strings of length seven contain two consecutive 0's?

Solution

a) Let a_n be the number of bit strings containing two consecutive 0's. In order to construct such a bit string we could start with a 1 and follow with a bit string of length $n-1$ containing two consecutive 0's, or we could start with 01 and follow with a bit string of length $n-2$ containing to consecutive 0's, or we could start with 00 and follow with any bit string of length $n-2$. These cases are exhaustive and mutually exclusive, so $a_n = a_{n-1} + a_{n-2} + 2^{n-2}$ for $n \geq 2$.

b) The initial conditions are $a_0 = 0$ and $a_1 = 0$ since there are no strings of length 0 or 1 which contain two consecutive 0's.

c) $a_2 = a_1 + a_0 + 2^0 = 0 + 0 + 1 = 1$
$a_3 = a_2 + a_1 + 2^1 = 1 + 0 + 2 = 3$
$a_4 = a_3 + a_2 + 2^2 = 3 + 1 + 4 = 8$
$a_5 = a_4 + a_3 + 2^3 = 8 + 3 + 8 = 19$
$a_6 = a_5 + a_4 + 2^4 = 19 + 8 + 16 = 43$
$a_7 = a_6 + a_5 + 2^5 = 43 + 19 + 32 = 94$
There are 94 bit strings of length 7 containing two consecutive 0's.

Question 3 Let $T(n) = 1 + T(\frac{n}{2})$ and $T(1) = 0$. Find a closed-form solution for $T(n)$ when n is a power of 2 and prove that it is correct.
Solution
Let's use plug-and-chug to find the closed form solution.
$T(n) = 1 + T(\frac{n}{2})$
$T(n) = 1 + 1 + T(\frac{n}{4})$
$T(n) = 1 + 1 + 1 + T(\frac{n}{8})$
From these we see a pattern, it appears that $T(n) = i + T(\frac{n}{2^i})$. *Let's use the definition one more time to check this guess:* $T(n) = i + 1 + T(\frac{n}{2^{i+1}})$, *as expected. Letting* $i = \log_2 n$ *in the equation for* $T(n)$ *we get that* $T(n) = \log_2 n + T(\frac{n}{2^{\log_2 n}}) = \log_2 n + T(\frac{n}{n}) = \log_2 n + T(1) = \log_2 n + 0 = \log_2 n$.
Let's verify that $T(n) = \log_2 n$ *with induction on* k *where* $n = 2^k$. *Let P(k) be that* $T(n) = \log_2 n$ *for* $n = 2^k$. *The base case is P(0) or that* $T(2^0) = T(1) = 0 = \log_2(2^0) = \log_2 1$ *which holds. Assume that* $T(n) = \log_2 n$ *for* $n = 2^k$. *Then* $T(2^{k+1}) = 1 + T(2^k) = 1 + \log_2 2^k = k + 1 = \log_2(2^{k+1})$, *as required.*

Question 4 Let $T(n) = 3T(n-1) + 4$ and $T(0) = 5$. Find a closed-form solution for $T(n)$ and prove that it is correct.
Hint: Remember that $\sum_{j=0}^{n} ar^j = a + ar + ar^2 + ... + ar^n = \frac{ar^{n+1} - a}{r-1}$.
Solution
Let's use plug-and-chug again to find the closed form solution.

$$
\begin{aligned}
T(n) &= 3T(n-1) + 4 & \\
T(n) &= 3(3T(n-2) + 4) + 4 & \textit{plug} \\
T(n) &= 3^2T(n-2) + 4 + 4 \cdot 3 & \textit{chug} \\
T(n) &= 3^2(3T(n-3) + 4) + 4 + 4 \cdot 3 & \textit{plug} \\
T(n) &= 3^3T(n-3) + 4 + 4 \cdot 3 + 4 \cdot 3^2 & \textit{chug} \\
T(n) &= 3^3(3T(n-4) + 4) + 4 + 4 \cdot 3 + 4 \cdot 3^2 & \textit{plug} \\
T(n) &= 3^4T(n-4) + 4 + 4 \cdot 3 + 4 \cdot 3^2 + 4 \cdot 3^3 & \textit{chug}
\end{aligned}
$$

It looks like $T(n) = 3^i T(n-i) + 4 + 4 \cdot 3 + ... + 4 \cdot 3^{i-1}$. *Using the hint we see that*

$$\begin{aligned} 4 + 4 \cdot 3 + ... + 4 \cdot 3^{i-1} &= \frac{4 \cdot 3^i - 4}{3 - 2} \\ &= \frac{4 \cdot 3^i - 4}{2} \\ &= 2 \cdot 3^i - 2. \end{aligned}$$

Thus we have $T(n) = 3^i T(n-i) + 2 \cdot 3^i - 2$. *Let's check our guess by doing one more step:* $T(n) = 3^i(3T(n-i-1)+4) + 2 \cdot 3^i - 2 = 3^{i+1}T(n-(i+1)) + 4 \cdot 3^i + 2 \cdot 3^i - 2 = 3^{i+1}T(n-(i+1)) + 6 \cdot 3^i - 2 = 3^{i+1}T(n-(i+1)) + 2 \cdot 3^{i+1} - 2$, *as expected. Letting* $i = n$ *we get*

$$\begin{aligned} T(n) &= 3^n T(n-n) + 2 \cdot 3^n - 2 \\ &= 3^n T(0) + 2 \cdot 3^n - 2 \\ &= 5 \cdot 3^n + 2 \cdot 3^n - 2 \\ &= 7 \cdot 3^n - 2. \end{aligned}$$

Now we verify that $T(n) = 7 \cdot 3^n - 2$ *using induction. Let* $P(n)$ *be that* $T(n) = 7 \cdot 3^n - 2$. *The base case is P(0) or that* $T(0) = 5 = 7 \cdot 3^0 - 2 = 7 - 2$ *which is true. Assume P(k) or that* $T(k) = 7 \cdot 3^k - 2$.

$$\begin{aligned} T(k+1) &= 3T(k) + 4 \\ &= 3(7 \cdot 3^k - 2) + 4 \\ &= 7 \cdot 3^{k+1} - 6 + 4 \\ &= 7 \cdot 3^{k+1} - 2 \end{aligned}$$

Therefore, $P(k+1)$ *holds which completes the proof by induction that* $T(n) = 7 \cdot 3^n - 2$.