Math 4108, Algebra II
HW 2 — Due Feb 3, 2017 (Fri)

Read and think about the problems in Chapters 3 and 4 of the textbook (Howie). Turn in the following problems. Justify all your answers.

1. Let $K = \mathbb{Q}(\sqrt[3]{2})$. Find all elements $\alpha \in K$ such that $K = \mathbb{Q}(\alpha)$.

2. Let $K = \mathbb{Q}(\sqrt{3}, \sqrt{5})$.
   (a) Prove that $\sqrt{3} \notin \mathbb{Q}(\sqrt{5})$.
   (b) Find a basis of K over $\mathbb{Q}$.
   (c) Show that the only subfields of K are $\mathbb{Q}$, $\mathbb{Q}(\sqrt{3})$, $\mathbb{Q}(\sqrt{5})$, $\mathbb{Q}(\sqrt{15})$, and K itself.
   (d) Find the minimum polynomial of $\sqrt{3} + \sqrt{5}$ over $\mathbb{Q}$.

3. Find the multiplicative inverse of $1 + \sqrt{3} + \sqrt{5} + \sqrt{15}$ in $\mathbb{Q}(\sqrt{3}, \sqrt{5})$.

4. Show that $[\mathbb{Q}(\sqrt{5} + \sqrt[3]{2}) : \mathbb{Q}] = 6$.

5. Let L be a field, K be a subfield of L, and $a, b \in L$ be algebraic over K of degrees m and n respectively. Prove that if m and n are relatively prime, then $[K(a, b) : K] = mn$.

6. Let K be a field. Prove that the following conditions are equivalent.
   (a) Every polynomial in $K[x]$ of degree ≥ 1 has a root in K.
   (b) Every polynomial in $K[x]$ of degree ≥ 1 *splits* over K, that is, it factors as a product of linear polynomials.
   (c) Every irreducible polynomial in $K[x]$ has degree 1.
   (d) There is no algebraic extension of K except K itself.

   A field K is called *algebraically closed* if any (hence all) of the conditions above are satisfied.

7. Prove that an algebraically closed field must contain infinitely many elements.

8. Let $\overline{\mathbb{Q}}$ denote the field of algebraic numbers:

   $$\overline{\mathbb{Q}} = \{a \in \mathbb{C} : a \text{ is algebraic over } \mathbb{Q}\}.$$

   Show that $\overline{\mathbb{Q}}$ is algebraically closed. You can use the Fundamental Theorem of Algebra, which says that $\mathbb{C}$ is algebraically closed.

   Note: An extension L of a field K is called an *algebraic closure* of K if L is algebraically closed and is algebraic over F. This exercise shows that the algebraic numbers form an algebraic closure of the rational numbers.

9. (a) Find the minimum polynomial of $\cos\left(\frac{2\pi}{5}\right)$.
   (b) Prove that a regular pentagon is constructible.