Field Extensions

Let L be a field extension of a field K.

- An element $\alpha \in L$ is said to be **algebraic over K** if there is a polynomial $f \in K[x]$ such that $f(\alpha) = 0$.
- The extension $L : K$ is called **algebraic** if every element of L is algebraic over K.
- The extension $L : K$ is called **finite** if $[L : K]$ is finite.
  Recall that $[L : K]$ is the dimension of L as a vector space over K.

Theorem (proven last class) If an element $\alpha \in L$ is algebraic over K, there there is a unique irreducible monic polynomial $m(x) \in K[x]$ such that $K(\alpha) = K[\alpha] = K[x]/\langle m(x) \rangle$. In particular, $[K(\alpha) : K] = \deg(m(x))$.
This polynomial $m(x)$ is called the **minimum polynomial** of α over K.

Prove the following.

1. Prove that the map $\varphi : \mathbb{Q}(\sqrt{2}) \to \mathbb{Q}(\sqrt{2})$ given by $a + b\sqrt{2} \mapsto a - b\sqrt{2}$ is an automorphism.
2. Prove that $\mathbb{Q}(\sqrt[3]{2}) \cong \mathbb{Q}(\sqrt[3]{2}e^{\frac{2\pi i}{3}})$, which are subfields of $\mathbb{C}$.
3. An element $\alpha \in L$ is algebraic over K if and only if $[K(\alpha) : K]$ is finite.
4. Every finite extension is algebraic. (Do you think the converse is true?)
5. Let $M \supseteq L \supseteq K$ be fields. Let $\alpha \in M$. If α is algebraic over K, then it is algebraic over L. (But the minimum polynomial of α over K may be different from that over L. Give an example.)
6. If $\alpha, \beta \in L$ are both algebraic over K, then $\alpha \pm \beta$, $\alpha\beta$, $\alpha\beta^{-1}$ (if $\beta \neq 0$) are also algebraic over K. Conclude that the set of elements in L that are algebraic over K form a subfield of L.